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REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: AFOSR - TR- 77 - 0190
2. GOVT ACCESSION NO.:
3. RECIPIENT'S CATALOG NUMBER:
4. TITLE (and Subtitle): INTEGRAL FUNCTIONALS, NORMAL INTEGRANDS AND MEASURABLE SELECTIONS.
5. TYPE OF REPORT & PERIOD COVERED: Interim rept.
6. PERFORMING ORG. REPORT NUMBER:
7. AUTHOR(s): R. Tyrrell Rockafellar
8. CONTRACT OR GRANT NUMBER(s): AF-AFOSR 72-2269-72
9. PERFORMING ORGANIZATION NAME AND ADDRESS: University of Washington, Department of Mathematics, Seattle, Washington 98195
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61102F 2304/A6
11. CONTROLLING OFFICE NAME AND ADDRESS: Air Force Office of Scientific Research/NM, Bolling AFB, Washington, DC 20332
12. REPORT DATE: 1977
13. NUMBER OF PAGES: 51 53p.
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office):
15. SECURITY CLASS. (of this report): UNCLASSIFIED
15a. DECLASSIFICATION DOWNGRADING SCHEDULE:
16. DISTRIBUTION STATEMENT (of this Report): Approved for public release; distribution unlimited.
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report):
18. SUPPLEMENTARY NOTES:

370 275

19. KEY WORDS (Continue on reverse side if necessary and identify by block number): Variational problems, integral functionals, measurability

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A fundamental notion in many areas of mathematics, including optimal control, stochastic programming, and the study of partial differential equations, is that of an integral functional. By this is meant an expression of the form $I_f(x) = \int_S f(s,x(s))\mu(ds)$, $x \in X$, where X is a linear space of measurable functions defined on a measure space (S, A, μ) and having values in a linear space E. This paper provides a thorough treatment of the properties of such

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20 Abstract

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functionals in the case of $E = R^n$, including properties of continuity convexity and duality.

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SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

Published version: in Nonlinear Operators and the Calculus of Variations, Lucien Waelbroeck (ed.), Springer-Verlag series Lecture Notes in Math., 1976.

INTEGRAL FUNCTIONALS, NORMAL INTEGRANDS AND MEASURABLE SELECTIONS

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A fundamental notion in many areas of mathematics, including optimization, probability, variational problems, functional analysis and operator theory, is that of an integral functional. By this is meant an expression of the form

$$I_f(x) = \int_S f(s,x(s))\mu(ds), \qquad x \in X,$$

where X is a linear space of measurable functions defined on a measure space $(S,\mathcal{A},\mu)$ and having values in a linear space E. The function $f: S \times E \to \overline{R}$ is the associated integrand.

Classically, only finite integrands on $S \times R^n$ were studied, usually under the assumption that $f(s,x)$ was continuous in x and measurable in s (the Carathéodory condition). However, from the modern point of view it is essential to admit possibly infinite values for f and I_f, since it is in this way that important kinds of constraints can most efficiently be represented. Such integrands require a distinctly new theoretical approach, where questions of measurability and the existence of measurable selections are prominent and are reflected in a concept of "normality".

The purpose of these notes is to provide a relatively thorough treatment of the most common case in applications, that where $E = R^n$. While many of the results have extensions in one way or another beyond this case, as indicated to some extent in the text, these are often more complicated technically and may require further restrictions. For example, it is only for R^n that one presently knows how to develop a complete theory without assuming that the measurable space is complete, an assumption which appears to be awkward in some situations. In treating infinite-dimensional spaces E, there are the usual problems of the multiplicity of topologies and dualities which must be ironed out. It is desirable, therefore, to have available a full and consistent exposition of the details in the basic case of $E = R^n$, freeing one from the need to search for auxilliary results through sequences of papers with varying frameworks.

*This work was supported in part by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under AFOSR grant number 72-2269.

The material below is divided in three principle sections. First we present the theory of measurable closed-valued multifunctions. Equivalent properties, any of which could actually be used as the definition of measurability, are discussed, and the basic measurable selection theorem of Kuratowski and Ryll-Nardzewski is derived via a stronger theorem on the existence of Castaing representations. (The proof, which is given in full, is simpler for R^n than in the more general case usually seen in the literature.) Much effort is devoted to establishing convenient means of verifying that a multifunction is indeed measurable.

The second part applies the results on measurable multifunctions to the study of normal integrands, a concept originally introduced by the author [1] in a setting of convexity, but developed here in more general terms. Again the emphasis is on measurability questions and the manufacture of tools which make easier the verification of "normality". Normal integrands are also important in the generation of measurable multifunctions given by systems of constraints, subdifferential mappings, etc.

These technical developments come to fruition in the theory of integral functionals presented in the third section of the notes. It is here also that convex analysis comes more to the front of the stage. This is due to natural considerations of duality, which are always important in a setting of functional analysis, as well as deeper reasons related to Liapunov's theorem and involving the weak compactness of level sets of integral functionals.

For obvious reasons of space, the discussion is limited to integral functionals on decomposable function spaces, such as Lebesgue spaces. These are characterized by the validity of a fundamental result on the interchange of integration and minimization. The treatment of more general function spaces usually relies heavily on this, more basic theory, as for example the case of Banach spaces of continuous functions as developed in [2], or the spaces of differentiable functions encountered in variational problems (cf. [13], [15], [26], [32]). We have made no attempt to cover the many results in such directions.

1. Measurable Closed-Valued Multifunctions.

In everything that follows, S is an arbitrary nonempty set equipped with a σ-algebra $\mathcal{A}$; thus $(S,\mathcal{A})$ is a general measurable space subject only to the restriction that $S \in \mathcal{A}$. Elements of $\mathcal{A}$ are called measurable subsets of S.

A multifunction $\Gamma : S \to X$, where X is another set, is, like a function, best defined simply as a subset of $S \times X$. The set of all $x \in X$ such that $(s,x) \in \Gamma$ for a given $s \in S$ is denoted by $\Gamma(s)$. Unfortunately, this notation is ambiguous in the special case where Γ happens to be a function ($\Gamma(s) = \{x\}$ or $\Gamma(s) = x$?), and it is slightly troublesome in suggesting more generally that Γ can really be thought of as a mapping assigning to each $s \in S$ a subset $\Gamma(s)$ of X. It is true that Γ gives rise to such a mapping and is uniquely determined by it, but of course the two are not the same. The mapping $s \to \Gamma(s)$ corresponds, strictly speaking, to a subset of $S \times 2^X$, and thus the question of whether or not it is measurable, for example, is properly answered in terms of the usual theory of measurable functions and the choice of a measurability structure on the space 2^X. This is not the point of view we want to adopt, and so the distinction should be borne in mind.

Nevertheless, it is hard to be a purist on such matters without having a nuisance with basic ways of writing things (e.g. one could write $\Gamma[s]$ in place of $\Gamma(s)$, reserving the latter for the unique element of $\Gamma[s]$ when one exists, and the mapping $s \to \Gamma[s]$ could be denoted by $[\Gamma]$). In practice, no serious confusion arises even if technicalities are slightly abused in this respect.

We content ourselves with the following notation for multifunctions $\Gamma : S \to X$, which, if a little redundant, does serve to emphasize the setting:

$$\operatorname{dom} \Gamma = \{s \in S \mid \Gamma(s) \neq \emptyset\},$$
$$\operatorname{gph} \Gamma = \{(s,x) \mid x \in \Gamma(s)\},$$
$$\Gamma(T) = \cup_{s \in T} \Gamma(s).$$

Of course, $\operatorname{gph} \Gamma$ is really no different from what we have called Γ, and $\operatorname{dom} \Gamma$ is its projection on S. We shall denote by $\Gamma^{-1} : X \to S$ the multifunction obtained by reversing the pairs constituting Γ; thus

$$\Gamma^{-1}(x) = \{s \in S \mid x \in \Gamma(s)\},$$
$$\Gamma^{-1}(C) = \cup_{x \in C} \Gamma^{-1}(x) = \{s \in S \mid \Gamma(s) \cap C \neq \emptyset\}.$$

For the most part, we shall be concerned only with multifunctions $\Gamma : S \to R^n$ which are closed-valued, in the sense that $\Gamma(s)$ is a closed

subset of R^n for every $s \in S$. Such a multifunction is said to be <u>measurable</u> (relative to the σ-field A), if for each closed set $C \subset R^n$ the set $\Gamma^{-1}(C)$ is measurable (i.e. belongs to A).

This definition of measurability of multifunctions was first adopted in a general context by Castaing, who in his thesis [3] proved its equivalence with a number of other possible definitions. Many such equivalences, which are very useful to know, will be stated below. It is important to realize, however, that they break down when R^n is replaced by a more general space, or if Γ is not closed-valued, and just which property should then be called "measurability" is open to controversy. We want the reader to understand that the present definition may well be subject to revision in such cases.

Note that if Γ is actually single-valued (hence trivially closed-valued) and everywhere nonempty-valued, i.e. Γ is a function, measurability reduces to the usual concept.

It is obvious that Γ is measurable if it is constant: $\Gamma(s) \equiv D$ for [illegible] closed set D. Another fact worth recording is that if Γ is measurable, $D \subset R^n$ is closed and $T \subset S$ is measurable, then the multifunction Γ' defined by

$$\Gamma'(s) = \begin{cases} \Gamma(s) \cap D & \text{if } s \in T \\ \emptyset & \text{if } s \notin T \end{cases}$$

is measurable. Of course, the measurability of Γ implies the measurability of the set dom Γ, inasmuch as dom $\Gamma = \Gamma^{-1}(R^n)$.

In the result which follows, we denote by $\operatorname{dist}(z,C)$ the Euclidean distance of z from a closed set $C \subset R^n$:

$$\operatorname{dist}(z,C) = \min\{|z-x| \mid x \in C\},$$

where $|\cdot|$ is the Euclidean norm. (This is interpreted as $+\infty$ if $C = \emptyset$.)

1A. PROPOSITION. <u>For a closed-valued multifunction</u> $\Gamma : S \to R^n$, <u>the following properties are equivalent</u>:

(a) Γ <u>is measurable</u>;

(b) $\Gamma^{-1}(C)$ <u>is measurable for all open sets</u> C;

(c) $\Gamma^{-1}(C)$ <u>is measurable for all compact sets</u> C;

(d) $\Gamma^{-1}(C)$ <u>is measurable for all closed balls</u> C;

(e) $\operatorname{dist}(z,\Gamma(s))$ <u>is a measurable function of</u> $s \in S$ <u>for each</u> $z \in R^n$.

PROOF. (c) $\Rightarrow$ (a). Let C be any closed set in R^n. Then $C = \cup_{k=1}^{\infty} C_k$, where each C_k is compact, and hence

$$(1.1) \qquad \Gamma^{-1}(C) = \cup_{k=1}^{\infty} \Gamma^{-1}(C_k).$$

We have each $\Gamma^{-1}(C_k)$ measurable, hence so is $\Gamma^{-1}(C)$.

(a) $\Rightarrow$ (d). This is trivial.

(d) $\Rightarrow$ (b). Let C be open. Then $C = \cup_{k=1}^{\infty} C_k$, where each C_k is a closed ball. Thus (1.1) again holds with $\Gamma^{-1}(C_k)$ measurable, and we conclude $\Gamma^{-1}(C)$ is measurable.

(b) $\Rightarrow$ (c). Given a compact set C, let

$$C_k = \{z \in R^n | \operatorname{dist}(z,C) < k^{-1}\} \quad \text{for } k = 1,2,\ldots$$

Then C_k is open, cl C_k is compact, and $C_k \supset \operatorname{cl} C_{k+1}$. We have $\Gamma(s) \cap C_k \neq \emptyset$ for all k if and only if $\Gamma(s) \cap \operatorname{cl} C_k \neq \emptyset$ for all k, and since $\Gamma(s)$ is closed, the latter is equivalent by compactness to

$$\emptyset \neq \cap_{k=1}^{\infty} \Gamma(s) \cap \operatorname{cl} C_k = \Gamma(s) \cap C.$$

Therefore

$$\Gamma^{-1}(C) = \cap_{k=1}^{\infty} \Gamma^{-1}(C_k),$$

and since each $\Gamma^{-1}(C_k)$ is measurable by assumption, it follows that $\Gamma^{-1}(C)$ is measurable.

(d) $\Leftrightarrow$ (e). We have $\operatorname{dist}(z,\Gamma(s)) \leq \alpha$ if and only if $\Gamma(s)$ meets the ball $z+\alpha B$ (B = closed unit ball, $\alpha \geq 0$). Thus

$$(1.2) \qquad \{s | \operatorname{dist}(z,\Gamma(s)) \leq \alpha\} = \Gamma^{-1}(z+\alpha B).$$

Condition (e) means that all the sets of the form on the left in (1.2) are measurable, while (d) means all those on the right are measurable. Q.E.D.

1B. THEOREM. <u>For a closed-valued multifunction</u> $\Gamma: S \rightarrow R^n$, <u>the following conditions are equivalent</u>:

(a) Γ <u>is measurable</u>;

(b) (<u>Castaing representation</u>): dom Γ <u>is measurable, and there is a countable (or finite) family</u> $(x_i | i \in I)$ <u>of measurable functions</u> $x_i : \operatorname{dom} \Gamma \rightarrow R^n$, <u>such that</u>

$$(1.3) \qquad \Gamma(s) = \operatorname{cl}\{x_i(s) | i \in I\} \quad \underline{\text{for all}} \quad s \in \operatorname{dom}\Gamma;$$

(c) <u>There is a countable family</u> $(x_i | i \in I)$ <u>of measurable functions</u> $x_i : S \rightarrow R^n$, <u>such that</u>

$$(1.4) \qquad \{s \in S | x_i(s) \in \Gamma(s)\} \quad \underline{\text{is measurable for all}} \quad i \in I,$$

$$(1.5) \qquad \Gamma(s) \cap \{x_i(s) | i \in I\} \quad \underline{\text{is dense in}} \quad \Gamma(s) \quad \underline{\text{for all}} \quad s \in S.$$

PROOF: (b) $\Rightarrow$ (c). Trivial.

(c) $\Rightarrow$ (b). We can suppose without loss of generality that $I = \{1,2,\ldots\}$. Let the measurable sets in (1.4) be denoted by S_i. Then $\cup_{i=1}^{\infty} S_i = \mathrm{dom}\Gamma$ by (1.5), so $\mathrm{dom}\Gamma$ is measurable. Let ξ be the function which agrees with x_1 on S_1, with x_2 on $S_2 \setminus S_1$, with x_3 on $S_3 \setminus (S_1 \cup S_2)$, etc. Then ξ is measurable, and $\xi(s) \in \Gamma(s)$ for all $s \in \mathrm{dom}\Gamma$. Now for each i let x_i be the function which agrees with x_i on S_i and with ξ on $(\mathrm{dom}\Gamma)\setminus S_i$. The functions x_i are also measurable and satisfy $x_i(s) \in \Gamma(s)$ for all s. Moreover (1.5) implies

$$\Gamma(s) = \mathrm{cl}\{x_i(s) \mid i = 1,2,\ldots\} \quad \text{for all } s \in \mathrm{dom}\, S.$$

In other words, (b) holds for the collection $(x_i \; i = 1,2,\ldots)$.

(b) $\Rightarrow$ (a). For any open set C, we have by (1.3) that

$$\Gamma^{-1}(C) = \cup_{i \in I}\{s \in \mathrm{dom}\Gamma \mid x_i(s) \in C\},$$

and hence $\Gamma^{-1}(C)$ is a countable union of measurable sets. Thus $\Gamma^{-1}(C)$ is measurable for all open sets C, and from Proposition 1A we see that Γ is measurable.

(a) $\Rightarrow$ (b). For every nonempty *closed* set $C \subset R^n$ and every $z \in R^n$, let

$$P_z C = \{x \in C \mid \mathrm{dist}(z,x) = \mathrm{dist}(z,C)\}.$$

(a nonempty compact set). Observe that if the points $z_0, z_1, \ldots, z_n$ of R^n are affinely independent (i.e. not contained in a hyperplane), then the set $P_{z_0} P_{z_1} \cdots P_{z_n} C$ consists of a single element. This follows from the fact that the set in question is contained in the intersection of a family of n-spheres with centers $z_0, z_1, \ldots, z_n$; by an elementary induction argument, any nonempty intersection of $k+1$ n-spheres in R^n is, for some $m < n-k$, an m-sphere in an m-dimensional affine subset of R^n.

Let I be the countable index set consisting of all $i = (z_0, z_1, \ldots, z_n)$ such that $z_0, z_1, \ldots, z_n$ have rational coordinates and are affinely independent, and for such i and each $s \in \mathrm{dom}\Gamma$ let $x_i(s)$ be the unique element of $P_{z_0} P_{z_1} \cdots P_{z_n} \Gamma(s)$. In particular, $x_i(s)$ is one of the points of $\Gamma(s)$ nearest to z_n, and since z_n ranges over all "rational" points of R^n as i ranges over I, we see that (1.3) holds. Hence to obtain (b), it will suffice to show that $x_i(s)$ is measurable in s for each $i \in I$. But this will follow from showing that if Γ' is any multifunction of the form $\Gamma'(s) = P_z\Gamma(s)$,

where Γ is measurable and z is a fixed point in R^n, then Γ' is measurable.

To prove the latter, we introduce for $k = 1,2,\ldots,$ the multifunction $\Gamma_k:S \to R^n$ such that $\Gamma_k(s)$ consists of all $x \in R^n$ satisfying

$$\text{(1.6)} \qquad \operatorname{dist}(x,\Gamma(s)) < k^{-1} \text{ and } \operatorname{dist}(z,x) < \operatorname{dist}(z,\Gamma(s)) + k^{-1}.$$

Observe that $\Gamma_k(s)$ is open for all s (nonempty if and only if $s \in \operatorname{dom}\Gamma$). Let C be any closed set. The condition $C \cap P_z\Gamma(s) \neq \emptyset$ is obviously equivalent to $C \cap \Gamma_k(s) \neq \emptyset$ for all k, and hence we have

$$\text{(1.7)} \qquad (\Gamma')^{-1}(C) = \cap_{k=1}^{\infty}\Gamma_k^{-1}(C).$$

On the other hand, denoting by D any countable dense subset of C (which exists since R^n itself has a countable dense subset), we have by the open-valuedness of Γ_k that

$$\text{(1.8)} \qquad \Gamma_k^{-1}(C) = \Gamma_k^{-1}(D) = \cup_{x \in D}\Gamma_k^{-1}(x).$$

But every set of the form $\Gamma_k^{-1}(x)$ consists of the elements $s \in S$ satisfying (1.6) for fixed x and z and hence is measurable by virtue of the implication (a) $\Rightarrow$ (e) in Proposition 1A for Γ. In this way, (1.7) and (1.8) confirm the measurability of $(\Gamma')^{-1}(C)$ for arbitrary closed C. Thus Γ' is measurable. Q.E.D.

The important equivalence of (a) and (b) in Theorem 1B was first established by Castaing [3]. It is for this reason that we shall call a family $(x_i \mid i \in I)$ with the properties in (b) a Castaing representation of Γ. The existence of such representations provides a handy tool in many situations. The following fact, which is the focus of all the theory of measurable multifunctions presented here, is an immediate consequence.

1C. COROLLARY (*Theorem on Measurable Selections*). *If* $\Gamma:S \to R^n$ *is a measurable closed-valued multifunction, there exists at least one measurable selection, i.e., a function* $x:\operatorname{dom}\Gamma \to R^n$ *such that* $x(s) \in \Gamma(s)$ *for all* $s \in \operatorname{dom}\Gamma$.

This result may be credited to Kuratowski and Ryll-Nardzewski [4]; although Castaing arrived at it independently at about the same time, he did not publish any details until much later [3]. An earlier proof by Rokhlin [5] is now known to be invalid [29]. Actually, these citations all refer to a more general selection theorem than 1C, namely where R^n is replaced by an arbitrary separable complete metrizable space (Polish

space, in the Bourbaki terminology). Theorem 1B also remains true in this case [3], but the proof is more complicated because a finite number of nearest point "projections" P_{z_j} does not suffice, and the compactness arguments must be replaced by something involving Cauchy sequences.

Another consequence of Theorem 1B is a simpler condition for measurability in certain cases, generalizing a criterion of Rockafellar [6] for convex-valued multifunctions.

1D. COROLLARY. <u>Let</u> $\Gamma:S \to R^n$ <u>be a multifunction such that, for all</u> $s \in S$, $\Gamma(s) = \mathrm{cl}(\mathrm{int}\Gamma(s))$ <u>(as is true, for instance, if</u> $\Gamma(s)$ <u>is an n-dimensional closed convex set). Then</u> Γ <u>is measurable if and only if</u> $\Gamma^{-1}(x)$ <u>is measurable for every</u> $x \in R^n$.

PROOF. The necessity is trivial. For the sufficiency, let $\{a_i | i \in I\}$ be any countable, dense subset of R^n, and let $(x_i | i \in I)$ be the corresponding family of constant functions: $x_i(s) \equiv a_i$. If $\Gamma^{-1}(a_i)$ is measurable for every $i \in I$, we have condition (b) of Theorem 1B fulfilled, and hence Γ is measurable. Q.E.D.

The <u>completion</u> of the measurable space (S,A) is the measurable space $(S,\hat{A})$, $\hat{A}$ being the intersection of all the σ-algebras of the form A_μ, where μ is a nonnegative, σ-finite measure on A and A_μ consists of all μ-measurable sets (or equivalently, all sets of the form $T \Delta T_o$, where $T \in A$, T_o is a subset of a set of μ-measure zero in A, and Δ denotes symmetric difference). One says that (S,A) is <u>complete</u> if $\hat{A} = A$. It is elementary that $(S,\hat{A})$ is always complete. Moreover, (S,A) is complete if $A = A_\mu$ for some μ.

Thus for example, if S is a Borel subset of some Euclidean space and A is the algebra of Lebesgue sets in S, we have (S,A) complete. If instead A is the algebra of Borel sets in S, (S,A) is not complete but could, for many purposes, be replaced by its completion $(S,\hat{A})$, $\hat{A}$ being in this case the algebra of all <u>universally measurable</u> sets in S.

The most important property of complete measurable spaces (S,A), for present needs, is that <u>for each measurable set</u> T <u>in</u> $S \times R^n$ <u>the projection of</u> T <u>on</u> S <u>is measurable</u>. The measurability of T here means, of course, that T belongs to the σ-algebra in $S \times R^n$ generated by products of sets in A and <u>Borel</u> measurable subsets of R^n. For a recent proof of this projection theorem in the general case where R^n is replaced by any Suslin space, see Sainte-Beuve [7].

1E. THEOREM. <u>Let</u> $\Gamma:S \to R^n$ <u>be closed-valued. Then among the following properties the implications</u> (c) ⇒ (a) ⇒ (b) <u>are always valid, with</u>

full equivalence among the three if the measurable space (S,A) is complete:

(a) Γ is a measurable multifunction;

(b) gph Γ is an $A \otimes B$-measurable subset of $S \times R^n$ (where B is the algebra of Borel sets);

(c) $\Gamma^{-1}(C)$ is measurable for all Borel sets $C \subset R^n$.

PROOF. (c) $\Rightarrow$ (a). Trivial from the definition.

(a) $\Rightarrow$ (b). Let $\{a_i\}$ be a dense sequence in R^n, and for each i and $k = 1,2,\ldots$, let C_{ik} be the closed ball with center a_i and radius k^{-1}. We have $x \in \Gamma(s)$ if and only if for all k there exists i such that $x \in C_{ik}$ and $\Gamma(s) \cap C_{ik} \neq \emptyset$. But this says that

$$\text{gph}\Gamma = \cap_{k=1}^{\infty} \cup_{i=1}^{\infty} [\Gamma^{-1}(C_{ik}) \times C_{ik}].$$

Each of the sets $\Gamma^{-1}(C_{ik}) \times C_{ik}$ belongs to $A \times B$, so this formula tells us that gphΓ belongs to $A \otimes B$.

(b) $\Rightarrow$ (c), assuming (S,A) is complete. Let C be any Borel set in R^n. Then $(\text{gph}\Gamma) \cap (S \times C)$ belongs to $A \times B$, and hence the projection of this set, which is just $\Gamma^{-1}(C)$, is measurable by the fact cited just prior to the theorem. Q.E.D.

For the more general context of R^n replaced by an infinite-dimensional space X, the $A \otimes B$-measurability of Γ can usefully be adopted as the definition of the measurability of Γ as a multifunction, and many of the facts developed here remain true. For a summary of some of the central aspects of this approach as regards integrands and integral functionals, see [8] and the references given there. However, it must be realized that the completeness of the measurable space is essential in such a context. This completeness may not always be convenient, as for instance in cases where one must deal with a whole family of Borel measures and perform frequent manipulations on the measurable spaces (such as taking products, which does not preserve completeness).

The next theorem, due essentially to Castaing [3], exploits the extra structure present if S is a Borel set in R^m. Recall that if S is a topological space, a multifunction $\Gamma: S \to R^n$ is said to be upper semicontinuous (or of closed graph) if gphΓ is closed in the product topology. (This is equivalent to the condition that for every compact set $C \subset R^n$, $\Gamma^{-1}(C)$ is closed.) On the other hand, Γ is said to be lower semicontinuous if for every open set $C \subset R^n$, $\Gamma^{-1}(C)$ is open. If both upper and lower semicontinuity are present, Γ is said to be continuous.

THEOREM 1F. Let S be a Borel subset of some Euclidean space, with A

the algebra of Lebesgue sets. Let $\Gamma : S \to R^n$ *be nonempty-closed-valued. Then the following conditions are equivalent:*

(a) Γ *is measurable;*

(b) *There is a closed-valued multifunction* $\Gamma' : S \to R^m$, *such that* $\mathrm{gph}\Gamma'$ *is a Borel set in* $S \times R^n$, *and* $\Gamma(s) = \Gamma'(s)$ *for almost every* s.

(c) (*Lusin property*). *For every* $\varepsilon > 0$ *there is a closed set* $T_\varepsilon \subset S$ *with* $\mathrm{mes}(S \setminus T_\varepsilon) < \varepsilon$, *such that* Γ *is continuous relative to* T_ε.

(d) *For every* $\varepsilon > 0$, *there is a closed set* $T_\varepsilon \subset S$ *with* $\mathrm{mes}(S \setminus T_\varepsilon) < \varepsilon$, *such that the set* $\{(s,x) | s \in T_\varepsilon,\ x \in \Gamma(s)\}$ *is closed.*

PROOF. (c) $\Rightarrow$ (d). Trivial.

(d) $\Rightarrow$ (b). For $\varepsilon = k^{-1}$, $k = 1,2,\ldots,$ let T be the union of the corresponding sequence of sets T_ε. Then T is measurable with $\mathrm{mes}(S \setminus T) = 0$, and the set $\{(s,x) | s \in T,\ x \in \Gamma(s)\}$ is a union of closed sets, hence Borel measurable. Thus (b) is satisfied with Γ' the restriction of Γ to T.

(b) $\Rightarrow$ (a). We have Γ' measurable by Theorem 1E, because (S, A) is complete. Let $C \subset R^n$ be closed. Then $(\Gamma')^{-1}(C)$ is measurable and differs from $\Gamma^{-1}(C)$ by at most a set of measure zero. Hence $\Gamma^{-1}(C)$ is measurable, and it follows that Γ is measurable.

(a) $\Rightarrow$ (c). First we demonstrate the argument can be reduced to the case where $\mathrm{mes}\, S < \infty$. Since S is a Borel set in R^m for some m, we can express it as the union of the disjoint Borel sets

$$S^k = \{s \in S | k-1 \le |s| < k\}, \quad k = 1,2,\ldots,$$

and these have $\mathrm{mes}\, S^k < \infty$. If (c) holds relative to every S^k, we can find for any $\varepsilon > 0$ a sequence of compact sets $T^k \subset S^k$, such that Γ is continuous relative to T^k and $\mathrm{mes}(S^k \setminus T^k) \le \varepsilon 2^{-k}$. No more than finitely many of the disjoint sets T^k touch any bounded region. Therefore, Γ is also continuous relative to $T = \cup_{k=1}^{\infty} T_k$, which is a closed subset of S with $\mathrm{mes}(S \setminus T) < \infty$.

In the rest of the proof, we assume $\mathrm{mes}\, S < \infty$. Let $(x_i | i=1,2,\ldots)$ be a Castaing representation of Γ. Let $\varepsilon > 0$. For each i, there exists by the usual form of Lusin's Theorem for measurable functions a compact set T_ε^i, such that x_i is continuous relative to T_ε^i, and $\mathrm{mes}(S \setminus T_\varepsilon^i) \le \varepsilon 2^{-i}$. Let $T_\varepsilon = \cap_{i=1}^{\infty} T_\varepsilon^i$. Then every x_i is continuous relative to T_ε, and T_ε is a compact set with $\mathrm{mes}(S \setminus T_\varepsilon) \le \varepsilon$. If $C \subset R^n$ is open, we have $x_i^{-1}(C) \cap T_\varepsilon$ open relative to T_ε for all i, so that the set

$$\Gamma^{-1}(C) \cap T_\varepsilon = \cup_{i=1}^{\infty} [x_i^{-1}(C) \cap T_\varepsilon]$$

is open relative to T_ε. Thus Γ is lower semicontinuous relative

to T_ε.

It remains only to show (assuming $\text{mes}\, S < \infty$) that (a) implies (d). Let $(C_k | k=1,2,\ldots)$ be an enumeration of all the (countably many) closed subsets of R^n complementary to the open balls U_k with rational center and radius. For each s, $\Gamma(s)$ is then the intersection of all the sets C_k containing it. Let $S_k = \Gamma^{-1}(U_k)$ and

$$S_k' = S \backslash S_k = \{s | \Gamma(s) \subset C_k\}.$$

Then S_k and S_k' are measurable (by criterion (c) in Proposition 1A), and

$$\text{gph}\Gamma = \cap_{k=1}^{\infty}[(S_k \times R^n) \cup (S_k' \times C_k)].$$

Fix $\varepsilon > 0$. For each k, there exist compact sets $K_k \subset S_k$ and $K_k' \subset S_k'$, such that

$$\text{mes}(S \backslash (K_k \cup K_k')) \leq \varepsilon 2^{-k}.$$

Let

$$T_\varepsilon = \cap_{k=1}^{\infty}(K_k \cup K_k').$$

Then T_ε is a compact set with $\text{mes}(S\backslash T_\varepsilon) \leq \varepsilon$, and we have

$$\{(s,x) | s \in T_\varepsilon, x \in \Gamma(s)\} = \cap_{k=1}^{\infty}[(K_k \times R^n) \cup (K_k' \times C_k)].$$

The latter set is closed, so (d) is established. Q.E.D.

The preceding results provide the main <u>direct</u> criteria for measurability that are convenient in practice. However, we add for completeness one further condition,which has been used as the definition of measurability by some authors, such as Debreu [9].

1G. PROPOSITION. <u>Let $\Gamma: S \to R^n$ be nonempty-compact-valued. Then Γ is a measurable multifunction if and only if the corresponding mapping from S to the space M, consisting of all compact subsets of R^n under the Hausdorff metric, is measurable (in the usual sense of functions from a measurable space to a metric space)</u>.

PROOF. Suppose first that this mapping from S to M is measurable. Let C be any closed subset of R^n, and let U be the open set in M consisting of all compact K such that $K \cap C = \emptyset$. By assumption, the set

$$\{s \in S | \Gamma(s) \in U\} = S \backslash \Gamma^{-1}(C)$$

is measurable, and therefore $\Gamma^{-1}(C)$ is measurable. Thus Γ is a measurable multifunction.

For the converse argument, let M_0 denote the collection of all finite sets in R^n consisting only of "rational" points. Then M_0 is countable and dense in M, so that every open set in M is the union of a countable family of closed balls whose centers belong to M_0. Therefore, to show the measurability of the mapping from S to M

associated with Γ, we need only verify that, for each such ball W, the set $\{s \in S \mid \Gamma(s) \in W\}$ is measurable. Suppose W has radius $\varepsilon > 0$ and center $F \in M_0$, and let B denote the closed unit ball in R^n. Then $K \in W$ if and only if $K \subset F + \varepsilon B$ and $F \subset K + \varepsilon B$, or in other words

$$K \cap (x + \varepsilon B) \neq \emptyset \quad \text{for each } x \in F,$$

$$K \cap (R^n \backslash (F + \varepsilon B)) = \emptyset.$$

It follows that the set $\{s \in S \mid \Gamma(s) \in W\}$ is the intersection of the finite family of sets $\Gamma^{-1}(x + \varepsilon B)$ for $x \in F$ (each of which is measurable by hypothesis) and

$$S \backslash \Gamma^{-1}(R^n \backslash (F + \varepsilon B)).$$

The latter is measurable by Proposition 1A, since $R^n \backslash (F + \varepsilon B)$ is open. Hence $\{s \in S \mid \Gamma(s) \in W\}$ is measurable. Q.E.D.

The chief goal of the theory of measurable multifunctions is to enable us to verify the existence of measurable selections for multifunctions Γ of the kinds that arise in practice, and this is to be accomplished by showing that Γ is measurable (cf. 1C). However, the criteria given above are not always easy to apply directly. Typically, Γ is given in terms of a more or less complicated construction involving other, simpler multifunctions, as well as certain functions ("integrands", which will be discussed in §2). The measurability properties of these more fundamental objects may be more accessible, and it is important to know how they are preserved under various operations. Without auxilliary results in this direction, no theorem on measurable selections can be viewed as more than a preliminary step towards applications. It may be remarked that the very choice of the definition of "measurability" is heavily influenced by such considerations; the appropriate category of multifunctions must not only possess measurable selections but also be convenient to manipulate.

The next series of results describes operations on closed-valued multifunctions that preserve measurability. The picture will be completed in §2 by analogous results about "integrands" and their intimate relation to multifunctions.

1H. PROPOSITION. *Let $\Gamma: S \to R^n$ be a closed-valued multifunction, and let Γ' be the multifunction such that, for each $s \in S$, $\Gamma'(s) = \text{cl co}\Gamma(s)$ (closed convex hull). Then Γ' is measurable (and closed-valued).*

The same is true if, in place of cl co$\Gamma(s)$, one takes the smallest closed cone containing $\Gamma(s)$, or the affine hull of $\Gamma(s)$, or the subspace generated by $\Gamma(s)$.

PROOF. We exploit the fact that every element of $\mathrm{co}\Gamma(s)$ can be expressed as a convex combination of $n+1$ (or fewer) elements of $\Gamma(s)$ (Carathéodory's Theorem). Let $(x_i \mid i \in I)$ be a Castaing representation of Γ (cf. comments following the proof of Theorem 1B). Let Λ be the set of all rational $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n)$ in R^{n+1}, such that $\lambda_k \geq 0$ and $\Sigma_{k=0}^{n} \lambda_k = 1$. For each of the countably many indices

$$j = (\lambda, i_0, \ldots, i_n) \in J = \Lambda \times I \times \cdots \times I \text{ (n+1 times)},$$

define the function x_j: dom$\Gamma \to R^n$ by

$$x_j(s) = \lambda_0 x_{i_0}(s) + \cdots + \lambda_n x_{i_n}(s).$$

Then $(x_j \mid j \in J)$ is a Castaing representation of Γ', and hence Γ' is measurable by Theorem 1B. (The proofs for the other cases in the proposition are analogous.) Q.E.D.

1I. PROPOSITION. <u>Let</u> $\Gamma_j: S \to R^{n_j}$ <u>be closed-valued and measurable for</u> $j = 1, \ldots, m$, <u>and for</u> $R^n = R^{n_1} \times \cdots \times R^{n_m}$ <u>let</u> $\Gamma: S \to R^n$ <u>be defined by</u>

$$\Gamma(s) = \Gamma_1(s) \times \cdots \times \Gamma_m(s).$$

<u>Then</u> Γ <u>is measurable (closed-valued).</u>

PROOF. Let $(x_i \mid i \in I_j)$ be a Castaing representation of Γ_j for $j = 1, \ldots, m$. For each of the countably many indices

$$j = (i_1, \ldots, i_m) \in J = I_1 \times \cdots \times I_m,$$

let $x_j = (x_{i_1}, \ldots, x_{i_m})$. Then $(x_j \mid j \in J)$ is a Castaing representation of Γ, so Γ is measurable. Q.E.D.

1J. PROPOSITION. <u>Let</u> $\Gamma_j: S \to R^n$ <u>be closed-valued and measurable for</u> $j = 1, \ldots, m$, <u>and let</u> $\Gamma: S \to R^n$ <u>be defined by</u>

$$\Gamma(s) = \mathrm{cl}(\Gamma_1(s) + \cdots + \Gamma_m(s)).$$

<u>Then</u> Γ <u>is measurable (closed-valued).</u>

PROOF. The argument is similar to that for 1I.

1K. COROLLARY. <u>Let</u> $\Gamma: S \to R^n$ <u>be a measurable closed-valued multifunction, and let</u> $a: S \to R^n$ <u>be a measurable function. Then the multifunction</u> Γ' <u>given by</u> $\Gamma'(s) = \Gamma(s) + a(s)$ <u>(translate) is measurable (closed-valued).</u>

1L. PROPOSITION. <u>Let</u> $\Gamma_i: S \to R^n$ <u>be closed-valued and measurable for each</u> $i \in I$ <u>(countable index set), and let</u> $\Gamma: S \to R^n$ <u>be defined by</u>

$$\Gamma(s) = \mathrm{cl}\cup_{i \in I} \Gamma_i(s).$$

<u>Then</u> Γ <u>is measurable (closed-valued).</u>

PROOF. For each open set $C \subset R^n$, we have

$$\Gamma^{-1}(C) = \cap_{i \in I} \Gamma_i^{-1}(C).$$

Hence by the equivalence of (a) and (b) in 1A, Γ is measurable. (The result also follows immediately via Castaing representations.) Q.E.D.

1M. THEOREM. *Let* $\Gamma_i : S \to R^n$ *be closed-valued and measurable for each* $i \in I$ *(countable index set), and let* $\Gamma : S \to R^n$ *be defined by*

$$\Gamma\,(s) = \cap_{i \in I} \Gamma_i(s).$$

Then Γ *is measurable (closed-valued). In particular, the set*

$$\{s \in S \mid \cap_{i \in I} \Gamma_i(s) \neq \emptyset\} = \operatorname{dom} \Gamma$$

is measurable.

PROOF. First we treat the case where $I = \{1,2\}$. Fix any closed set C, and define the closed-valued multifunctions Γ_1' and Γ_2' by

$$\Gamma_1'(s) = C \cap \Gamma_1(s), \quad \Gamma_2'(s) = -\Gamma_2(s).$$

Then Γ_1' and Γ_2' are measurable, and one has

$$C \cap \Gamma_1(s) \cap \Gamma_2(s) \neq \emptyset \iff 0 \in \Gamma_1'(s) + \Gamma_2'(s).$$

Therefore

$$\Gamma^{-1}(C) = (\Gamma_1' + \Gamma_2')^{-1}(0),$$

and we may conclude via Proposition 1J that $\Gamma^{-1}(C)$ is measurable. Thus Γ is measurable.

The validity of the theorem for $I = \{1,2\}$ implies by induction its validity for any finite I. It remains to consider the case where I is infinite; we can suppose $I = \{1,2,\ldots\}$. For each index k, the closed-valued multifunction Γ_k defined by

$$\Gamma_k(s) = \cap_{i=1}^{k} \Gamma_i(s)$$

is measurable by what has already been proved. For each compact set $C \subset R^n$, we have $\Gamma(s) \cap C \neq \emptyset$ if and only if $\Gamma_k(s) \neq C$ for all k. Therefore

$$\Gamma^{-1}(C) = \cap_{k=1}^{\infty} \Gamma_k^{-1}(C),$$

where $\Gamma_k^{-1}(C)$ is measurable, and it follows that $\Gamma^{-1}(C)$ is measurable. This establishes the measurability of Γ by way of criterion (c) of Proposition 1A. Q.E.D.

Theorem 1M, a crucial fact in several arguments below, was first proved in the present framework in Rockafellar [6]. Of course, if the measurable space is complete, the result is trivial in terms of criterion (b) of Theorem 1E, and hence it is trivial also in general contexts

where this criterion is adopted as the definition of the measurability of a multifunction.

The next result is new, at least in the stated generality.

1N. THEOREM. *Let $\Gamma:S \to R^n$ be closed-valued and measurable, and for each $s \in S$ let $A_s:R^n \to R^m$ be a multifunction with closed graph depending measurably on s (i.e. the multifunction $G(s) = \text{gph } A_s$ is closed-valued and measurable). Then the multifunction $\Gamma':S \to R^m$ defined by*

$$\Gamma'(s) = \text{cl}A_s(\Gamma(s))$$

is measurable (closed-valued). (The closure operation here is superfluous if $\Gamma(s)$ is bounded.)

PROOF. Let C be any open set in R^n. Then C is the union of a sequence of closed sets C_k. For each k, define the multifunction $G_k:S \to R^n \times R^n$ by $G_k(x) = \Gamma(s) \times C_k$. Then G_k is measurable by 1I. Since C is open, we have

$$(\Gamma')^{-1}(C) = \{s | C \cap A_s(\Gamma(s)) \neq \emptyset\}$$

$$= \cup_{k=1}^{\infty}\{s| \ (x,y) \in \text{gph}A_s \text{ with } x \in \Gamma(s),\ y \in C_k\}$$

$$= \cup_{k=1}^{\infty}\{s|G(s) \cap G_k(s) \neq \emptyset\}.$$

Each of the sets in the latter union is measurable by Theorem 1M. Therefore $(\Gamma')^{-1}(C)$ is measurable, and we conclude from condition (b) of Proposition 1A that Γ' is measurable. (If $\Gamma(s)$ is bounded, it is compact, and one sees easily that $A_s(\Gamma(s))$ is closed, making the closure operation in the definition of $\Gamma'(s)$ unnecessary.) Q.E.D.

1P. COROLLARY. *Let $\Gamma:S \to R^n$ be closed-valued and measurable, and for each $s \in S$ let $F:S \times R^n \to R^m$ be a mapping such that $F(s,x)$ is measurable in s and continuous in x. Let $\Gamma':S \to R^m$ be defined by*

$$\Gamma'(s) = \text{cl}F(s,\Gamma(s)).$$

Then Γ' is measurable (closed-valued).

PROOF. Let $A_s = F(s,\cdot)$. Let $(a_i|i \in I)$ be a countable dense subset of R^n. For each i define $z_i:S \to R^n \times R^m$ by $z_i(s) = (a_i,F(s,a_i))$. Then $\{z_i|i \in I\}$ is a Castaing representation for the multifunction $G(s) = \text{gph}A_s$, which therefore is measurable (Theorem 1B). Thus the hypothesis of Theorem 1N is satisfied. Q.E.D.

1Q. COROLLARY. *Let $\Gamma:S \to R^n$ be closed-valued and measurable, and for each $s \in S$ let $F:S \times R^m \to R^n$ be a mapping such that $F(s,u)$ is measurable in s and continuous in u. Let $\Gamma':S \to R^m$ be defined by*

$$\Gamma'(s) = \{u \in R^m \mid F(s,u) \in \Gamma(s)\}.$$

Then Γ' _is measurable_ (_closed-valued_).

PROOF. Clearly $\Gamma'(s)$ is closed for all s. Let $A_s = F(s,\cdot)^{-1}$. By an argument similar to the one in the preceding corollary, the multifunction $G(s) = \text{gph } A_s$ has a Castaing representation, and hence Theorem 1N is applicable. Q.E.D.

1R. COROLLARY. _Let_ $\Gamma: S \to R^m \times R^n \times R^k$ _be a measurable_, _closed-valued multifunction_, _and let_ $\Gamma': S \to R^n$ _be defined by_

$$\Gamma'(s) = \text{cl}\{x \mid \exists w \in R^m \text{ with } (w,x,u(s)) \in \Gamma(s)\},$$

where $u: S \to R^k$ _is measurable_. _Then_ Γ' _is measurable_ (_closed-valued_). (_The closure operation here is superfluous if_ $\Gamma(s)$ _is bounded_.)

PROOF. Let F_1 be the projection $(w,x) \to x$, and let $F_2(s,w,x) = (w,x,u(s))$. Let

$$\Gamma''(s) = \{(w,x) \mid F_2(s,w,x) \in \Gamma(s)\}.$$

Then Γ'' is measurable by 1Q, and $\Gamma'(s) = \text{cl } F_1(\Gamma''(s))$, so that Γ' is measurable by 1P. Q.E.D.

REMARK. Two new articles will be especially useful to those in need of a more general theory of measurable multifunctions than is furnished here. Wagner [29] has put together a comprehensive survey of the existing literature. Delode, Arino and Penot [30] have worked out a new and broader framework for the subject, from the point of view of fiber spaces, and have thereby obtained extensions of a number of previous results, for example, involving a weakening of the "completeness" requirement in Theorem 1E.

2. Normal Integrands.

For present purposes, any function $f: S \times R^n \to \overline{R}$ will be called an integrand on $S \times R^n$. Here $\overline{R}$ denotes the extended reals: $\overline{R} = R \cup \{\underline{+}\infty\}$. Corresponding to f and completely determining it, is its epigraph multifunction $E_f: S \to R^{n+1}$, defined by

$$(2.1) \qquad E_f(s) = \text{epi } f(s,\cdot) = \{(x,\alpha) \in R^n \times R \mid \alpha \geq f(s,x)\}.$$

We shall say that f is a lower semicontinuous integrand if $f(s,x)$ is l.s.c. (lower semicontinuous) in x for each s (i.e., E_f is closed-valued), and that f is a normal integrand if, besides this, E_f is a measurable multifunction. Of course, normality depends on the choice of the σ-algebra A; if more than one choice is possible, one can speak of f being A-normal, for clarity.

It is convenient to say that the function $f(s,\cdot)$ is proper on R^n if $f(s,x) > -\infty$ for all x and $f(s,x) \not\equiv +\infty$, and to call f a proper integrand if $f(s,\cdot)$ is proper in this sense for every $s \in S$. Furthermore, f is said to be a convex integrand if $f(s,x)$ is convex in x for each s, i.e., if E_f is convex-valued. Thus, for a proper integrand, $f(s,\cdot)$ is obtained for each s by extending as $+\infty$ a certain finite function defined on a nonempty set

$$(2.2) \qquad \text{dom } f(s,\cdot) = \{x \in R^n \mid f(s,x) < +\infty\}.$$

This set is convex for all s, if f is a convex integrand. Observe that the multifunction $s \to \text{cl dom } f(s,\cdot)$ is measurable if f is normal, since $\text{dom } f(s,\cdot)$ is just the image of $E_f(s)$ under the projection $F:(x,\alpha) \to x$ (Corollary 1P).

The theory of normal integrands with possibly infinite values was introduced and developed by Rockafellar in a series of papers [1], [2], [6], [8], [10], that originally treated only the convex case. A different definition of normality, taking advantage of convexity, was employed in most of this work, but it agrees with the present definition applied to convex integrands, as will be seen below. However, there is one slight change of terminology to be noted: what previously was a normal convex integrand is now a proper normal convex integrand.

The classical precursors of normal integrands are finite integrands satisfying the Carathéodory conditions. These will be shown to fit in as a special case.

Various results, generalizing some of the development in these notes to spaces other than R^n, may be found in [8] and, more recently, in Valadier [11], Castaing [24] and Delode-Arino-Penot [30].

Obviously, any integrand of the form $f(s,x) \equiv \phi(x)$, where $\phi: R^n \to \overline{R}$ is lower semicontinuous, is normal. The following results

furnish other criteria.

2A. THEOREM. Let f be a lower semicontinuous integrand on $S \times R^n$. If f is normal, then f is $\mathcal{A} \otimes \mathcal{B}$-measurable (where $\mathcal{B}$ is the algebra of Borel sets). The converse is true if the measurable space $(S,\mathcal{A})$ is complete.

PROOF. Necessity. For $\beta \in R$, define $\Gamma_\beta: S \to R^n$ by

$$\Gamma_\beta(s) = \{x \mid f(s,x) \le \beta\}.$$

Then Γ_β is closed-valued. For every closed $C \subset R^n$, we have $\Gamma_\beta^{-1}(C) = E_f^{-1}(C_\beta)$, where

$$C_\beta = \{(x,\alpha) \in R^{n+1} \mid x \in C, \alpha = \beta\},$$

and since E_f is a measurable multifunction, this implies $\Gamma_\beta^{-1}(C)$ is measurable. Thus Γ_β is measurable, and it follows from Theorem 1E that the set

$$\mathrm{gph}\Gamma_\beta = \{(s,x) \mid f(s,x) \le \beta\}$$

is $\mathcal{A} \otimes \mathcal{B}$-measurable. This being true for every $\beta \in R$, f is $\mathcal{A} \otimes \mathcal{B}$-measurable.

Sufficiency. If f is $\mathcal{A} \otimes \mathcal{B}$-measurable, then so is the function $g(s,x,\alpha) = f(s,x)-\alpha$ on $S \times R^{n+1}$. This implies the $\mathcal{A} \otimes \mathcal{B}$-measurability of the set

$$\{(s,x,\alpha) \mid g(s,x,\alpha) \le 0\} = \mathrm{gph}\, E_f.$$

Assuming $(S,\mathcal{A})$ to be complete, we can conclude from Theorem 1E that E_f is a measurable multifunction, i.e., f is normal. Q.E.D.

2B. COROLLARY. If f is a normal integrand on $S \times R^n$, and $x: S \to R^n$ is a measurable function, then the function $s \to f(s,x(s))$ is measurable.

PROOF. The transformation $\xi: s \to (s,x(s))$ is measurable from $(S,\mathcal{A})$ to $(S \times R^n, \mathcal{A} \otimes \mathcal{B})$. (For all sets T in $\mathcal{A} \times \mathcal{B}$, $\xi^{-1}(T)$ is measurable, and hence the same must be true for T in the σ-algebra $\mathcal{A} \otimes \mathcal{B}$ generated by $\mathcal{A} \times \mathcal{B}$.) We know from Theorem 2A that f is a measurable function with respect to $\mathcal{A} \otimes \mathcal{B}$, and therefore $f\xi$ is measurable. Q.E.D.

As with measurable multifunctions, the $\mathcal{A} \otimes \mathcal{B}$-measurability property can be adopted as the definition of the normality of an integrand when the measurable space $(S,\mathcal{A})$ is complete. This approach then allows an easy extension of much of the theory below to cases where R^n is replaced by an infinite-dimensional space; cf. [8].

2C. PROPOSITION. For an integrand f on $S \times R^n$, the following conditions are equivalent:

(a) both f and $-f$ are normal and proper;

(b) (Carathéodory condition): $f(s,x)$ is finite, measurable in s, and continuous in x.

PROOF. (a) $\Rightarrow$ (b). For each fixed s, neither the function $f(s,\cdot)$ nor $-f(s,\cdot)$ takes on the value $+\infty$, and both are lower semicontinuous. Therefore, $f(s,x)$ is finite and continuous in x. On the other hand, $f(s,x)$ is measurable in s for each fixed s by 2B.

(b) $\Rightarrow$ (a). Let D and P be countable dense subsets of R^n and R_+, respectively. For each $j = (a,\beta)$ in $J = D \times P$ define $y_j\colon S \to R^{n+1}$ by

$$y_j(s) = f(s,a) + \beta.$$

Then $(y_j \mid j \in J)$ is a Castaing representation for E_f, so by Theorem 1B we have E_f measurable (i.e., f normal). Q.E.D.

We shall call f a Carathéodory integrand if it has property (b) in 2C; thus Carathéodory integrands are examples of normal integrands. In fact, they are among the most important in their own right and in the construction of more general normal integrands.

More generally, we shall call a function $F\colon S \times R^n \to R^m$ a Carathéodory mapping, if $F(s,x)$ is measurable in s and continuous in x. Such mappings have already been encountered in 1P and 1Q.

The next result, for convex integrands, ties the present concept of normality in with the measurability property originally used to define normality in [1].

2D. PROPOSITION. Let f be a lower semicontinuous, convex integrand on $S \times R^n$. Then f is normal if and only if there is a countable family $(x_i \mid i \in I)$ of measurable functions $x_i\colon S \to R^n$, such that

(i) $f(s,x_i(s))$ is measurable in s for each $i \in I$,

(ii) $\{x_i(s) \mid i \in I\} \cap \operatorname{dom} f(s,\cdot)$ is dense in $\operatorname{dom} f(s,\cdot)$ for each $s \in S$.

PROOF. Necessity. If f is normal, the multifunction E_f has a Castaing representation $(y_i \mid i \in I)$ by Theorem 1B, and each y_i is of the form $y_i(s) = (x_i(s),\alpha_i(s))$, where $x_i\colon S \to R^n$ is measurable. Then (ii) holds trivially, because $\operatorname{dom} f(s,\cdot)$ (defined in (2.2)) is just the projection of $E_f(s)$ on R^n, while on the other hand (i) holds by 2B.

Sufficiency. Here we use the fact that, by convexity, any dense subset $D(s)$ of $\operatorname{dom} f(s,\cdot)$ yields

$$E_f(s) = \operatorname{cl}\{(x,\alpha) \in R^{n+1} \mid x \in D(s), \alpha \geq f(s,x)\}$$

[12,§7]. Given a family $(x_i \mid i \in I)$ with the properties in question,

let Q be a countable dense subset of R, and define the family $(y_j | j \in J)$ for $j = (i,\alpha)$ in $J = I \times Q$ as follows: $y_j(s) = (x_i(s),\alpha)$. Then y_j is measurable, and for each $s \in S$ we have

$$E_f(s) = \mathrm{cl}[E_f(s) \cap \{y_j(s) | j \in J\}]$$

by (ii). At the same time, for each $j \in J$ the set

$$\{s | y_j(s) \in E_f(s)\} = \{s | f(s,x_i(s)) \leq \alpha\}$$

is measurable by (i). Thus condition (c) of Theorem 1B is satisfied by E_f and $\{y_j | j \in J\}$, which allows us to conclude that E_f is measurable. Q.E.D.

2E. COROLLARY. *Let f be a lower semicontinuous, convex integrand on $S \times R^n$ such that* $\mathrm{dom}\, f(s,\cdot)$ *has a nonempty interior for every s. Then f is normal if and only if $f(s,x)$ is measurable with respect to s for each x.*

PROOF. Sufficiency follows from Proposition 2D, by taking $(x_i | i \in I)$ to be a family of constant functions with values in a dense subset of R^n. Necessity is immediate from Corollary 2B.

The equivalence of (b) and (c) in the next theorem was proved by Ekeland and Temam [13,p.216], who adopted (b) as their definition of normality (with the slight difference that they required $f(s,x)$ to be lower semicontinuous in x only for almost every s).

2F. THEOREM. *Let S be a Borel subset of some Euclidean space, with A the algebra of Lebesgue sets. Let f be any lower semicontinuous integrand on $S \times R^n$. Then the following conditions are equivalent:*

(a) *f is a normal integrand.*

(b) *There is a Borel measurable function $g: S \times R^n \to \bar{R}$ such that, for almost every $s \in S$, $f(s,x) = g(s,x)$ for all $x \in R^n$.*

(c) *For every $\varepsilon > 0$, there is a closed set $T_\varepsilon \subset S$ with* $\mathrm{mes}(S \backslash T_\varepsilon) < \varepsilon$, *such that $f(s,x)$ is lower semicontinuous in (s,x) relative to $T_\varepsilon \times R^n$.*

PROOF. The implication (c) $\Rightarrow$ (b) is elementary, while Theorem 1F, applied to E_f, yields (b) $\Rightarrow$ (a) $\Rightarrow$ (c). Q.E.D.

2G. COROLLARY. *Let S be a Borel subset of some Euclidean space, with A the algebra of Lebesgue sets, and let f be a finite integrand on $S \times R^n$. Then the following properties are equivalent:*

(a) *f is a Carathéodory integrand;*

(b) (*Scorzà-Dragoni property*): *for every $\varepsilon > 0$, there is a closed set $T_\varepsilon \subset S$ with* $\mathrm{mes}(S \backslash T_\varepsilon) < \varepsilon$, *such that f is continuous*

relative to $T_\varepsilon \times R^n$.

PROOF. This is immediate from Theorem 2F and Proposition 2C. Q.E.D.

Corollary 2G is the well-known theorem of Scorzà-Dragoni. Part (b) of Theorem 2F complements Theorem 2A in the special case of a complete measurable space of the form in 2F.

Next on the agenda is a further elucidation of the relationship between integrands and multifunctions.

2H. PROPOSITION. *Let* ψ_Γ *be the indicator integrand of a multifunction* $\Gamma: S \to R^n$, *i.e.*

$$\psi_\Gamma(s,x) = \begin{cases} 0 & \text{if } x \in \Gamma(x), \\ +\infty & \text{if } x \notin \Gamma(x). \end{cases} \tag{2.4}$$

Then ψ_Γ *is a normal integrand if and only if* Γ *is a measurable closed-valued multifunction.*

PROOF. This is obvious from Proposition 1H and the representation $E_{\psi_\Gamma}(s) = \Gamma(s) \times R_+$. Q.E.D.

2I. PROPOSITION. *Let* $\Gamma: S \times R^n$ *be a multifunction of the form*

$$\Gamma(s) = \{x \mid f(s,x) \le \alpha(s)\},$$

where f *is a normal integrand on* $S \times R^n$ *(e.g., a Carathéodory integrand), and* $\alpha: S \to \overline{R}$ *is measurable. Then* Γ *is closed-valued and measurable.*

PROOF. Since $f(s,\cdot)$ is lower semicontinuous, $\Gamma(s)$ is closed. Let $A: S \to R$ be the closed-valued multifunction defined by $A(s) = \{\beta \in R \mid \beta \ge \alpha(s)\}$. Then A is measurable, because α is measurable. Considering an arbitrary closed set $C \subset R^n$, we define a corresponding multifunction $\Gamma': S \to R^{n+1}$ by $\Gamma'(s) = C \times A(s)$. Then Γ' is closed-valued and measurable (Proposition 1I). We have

$$\Gamma^{-1}(C) = \{s \mid \Gamma'(s) \cap E_f(s) \ne \emptyset\},$$

and the latter set is measurable by Theorem 1M. Thus $\Gamma^{-1}(C)$ is measurable for all closed C. Q.E.D.

Proposition 2I is important in providing, in conjunction with the above conditions for normality, especially 2A, 2C(b) and 2F(b), an easily recognizable class of measurable multifunctions to which the operations in the preceding section may be applied.

As an illustration, we have the following version of the famous result in optimal control originally known as Filippov's lemma.

2J. THEOREM. (<u>Implicit Measurable Functions</u>). <u>Let</u> $\Gamma: S \to R^n$ <u>be a multifunction of the general form</u>

$$\text{(2.5)}\quad \Gamma(s) = \{x \in C(s) \mid F(s,x) = a(s) \ \text{and}\ f_i(s,x) \le \alpha_i(s) \ \text{for all}\ i \in I\},$$

<u>where</u> $C: S \to R^n$ <u>is closed-valued and measurable</u>, $F: S \times R^n \to R^m$ <u>is a Carathéodory mapping</u>, $(f_i \mid i \in I)$ <u>is a countable collection of normal integrands (e.g., Carathéodory integrands) on</u> $S \times R^n$, $a: S \to R^m$ <u>is measurable, and</u> $\alpha_i: S \to \overline{R}$ <u>is measurable</u>.

<u>Then</u> Γ <u>is measurable (closed-valued), and hence</u> Γ <u>has a measurable selection where it is nonempty-valued (i.e., relative to</u> $\operatorname{dom}\Gamma$).

PROOF. Let

$$D(s) = \{x \in R^n \mid F(s,x) = a(s)\},$$

$$\Gamma_i(s) = \{x \in R^n \mid f_i(s,x) \le \alpha_i(s)\} \quad \text{for each } i \in I.$$

Then D and Γ_i are closed-valued and measurable (Corollary 1Q and Proposition 2I). We have

$$\Gamma(s) = C(s) \cap D(s) \cap_{i \in I} \Gamma_i(s),$$

and therefore Γ is measurable by Theorem 1M. A measurable selection relative to $\operatorname{dom}\Gamma$ then exists by 1C. Q.E.D.

For applications to optimization problems, it is useful to have the following complement to Theorem 2J.

2K. THEOREM. <u>Let</u> f <u>be a normal integrand on</u> $S \times R^n$, <u>and let</u> $\Gamma: S \to R^n$ <u>be a measurable, closed-valued multifunction (e.g.,</u> $\Gamma(s)$ <u>of form</u> (2.5), <u>or</u> $\Gamma(s) \equiv R^n$). <u>Then the function</u> $m: S \to \overline{R}$ <u>given by</u>

$$m(s) = \inf_{x \in \Gamma(s)} f(s,x)$$

<u>and the closed-valued multifunction</u> $M: S \to R^n$ <u>given by</u>

$$M(s) = \operatorname*{arg\,min}_{x \in \Gamma(s)} f(s,x)$$

<u>are both measurable</u>.

PROOF. To demonstrate the measurability of m, we consider the closed-valued multifunction $\Gamma': S \to R^{n+1}$ defined by

$$\Gamma'(s) = E_f(s) \cap [\Gamma(s) \times R].$$

This is measurable by 1M (and 1I). For any $\beta \in R$, we have

$$\{s \mid m(s) < \beta\} = (\Gamma')^{-1}(R^n \times (-\infty,\beta)),$$

which is a measurable set by property (b) of 1A. Hence m is measurable, and since

$$M(s) = \{x \in \Gamma(s) \mid f(s,x) \le m(s)\},$$

the measurability of M follows by Theorem 2J. ($M(s)$ is closed, because $f(s,\cdot)$ is lower semicontinuous.) Q.E.D.

We turn now to the methods for generating new normal integrands from given ones.

2L. PROPOSITION. <u>Let</u> f <u>be an integrand on</u> $S \times R^n$ <u>of the form</u>

$$f(s,x) = \sup_{i \in I} f_i(s,x),$$

<u>or instead</u>,

$$f(s,x) = \liminf_{x' \to x} \inf_{i \in I} f_i(s,x'),$$

<u>where</u> $(f_i | i \in I)$ <u>is a countable family of normal integrands. Then</u> f <u>is normal</u>.

PROOF. In the first case $E_f(s) = \cap_{i \in I} E_{f_i}(s)$, so the normality is immediate from Theorem 1M. In the second case, $E_f(s)$ is the closure of $\cup_{i \in I} E_{f_i}(s)$, and we can apply Proposition 1L. Q.E.D.

2M. PROPOSITION. <u>Let</u> f <u>be an integrand on</u> $S \times R^m$ <u>of the form</u>

$$f(s,x) = \Sigma_{i=1}^m f_i(s,x),$$

<u>where each</u> f_i <u>is a proper, normal integrand. Then</u> f <u>is normal</u>.

PROOF. It is sufficient to consider $m = 2$. Define $\Gamma: S \to R^{n+1} \times R^{n+1}$ by $\Gamma(s) = E_{f_1}(s) \times E_{f_2}(s)$, and $A: R^{n+1} \times R^{n+1} \to R^{n+1}$ by

$$A(x_1,\alpha_1,x_2,\alpha_2) = \begin{cases} (x_1,\alpha_1+\alpha_2) & \text{if } x_2 = x_1 \\ \emptyset & \text{if } x_2 \neq x_1, \end{cases}$$

so that $E_f(s) = A(\Gamma(s))$. Here Γ is measurable by Proposition 1I, while A has closed graph. We have $E_f(s)$ closed (since $f(s,\cdot)$ inherits lower semicontinuity from $f_1(s,\cdot)$ and $f_2(s,\cdot)$, as is obvious from considering the "lim inf" at any point), and therefore E_f is measurable by Theorem 1N. Q.E.D.

Of course, some of the terms in the sum in Proposition 2M could be indicator integrands as in Proposition 2H (e.g., with Γ as in Theorem 2J).

2N. PROPOSITION. <u>Let</u> f <u>be an integrand on</u> $S \times R^n$ <u>of the form</u>

$$f(s,x) = \phi(s,g(s,x)), \tag{2.6}$$

<u>where</u> g <u>is a proper, normal integrand on</u> $S \times R^n$ <u>and</u> ϕ <u>is a normal integrand on</u> $S \times R$ <u>with</u> $\phi(s,\alpha)$ <u>nondecreasing in</u> α (<u>convention</u>: $\phi(s,+\infty) = +\infty$). <u>Then</u> f <u>is normal</u>.

<u>Similarly</u>, f <u>is normal if it is of the form</u> (2.6), <u>with</u> ϕ <u>a normal integrand on</u> $S \times R^m$ <u>and</u> g: $S \times R^n \to R^m$ <u>a Carathéodory mapping</u>.

PROOF. Obviously $f(s,x)$ is lower semicontinuous in x, so $E_f(s)$ is closed. Define $A\colon R^{n+1} \to R^{n+1}$ by

$$A_s(x,\alpha) = \{(x,\beta) \mid \beta \geq \phi(s,\alpha)\},$$

so that $E_f(s) = A_s(E_g(s))$. We have E_g closed-valued and measurable, because g is normal, while $\operatorname{gph} A_s$ is closed and measurable in s, because ϕ is normal. Hence E_f is measurable by Theorem 1N.

To prove the other assertion, let $F(s,x,\alpha) = (g(s,x),\alpha)$, so that $F\colon S \times R^{n+1} \to R^{m+1}$ is a Carathéodory mapping with

$$E_f(s) = \{(x,\alpha) \mid F(s,x,\alpha) \in E_\phi(s)\}.$$

The measurability of E_f then follows from Corollary 1Q. Q.E.D.

2P COROLLARY. *Let f be an integrand on $S \times R^n$ of the form*

$$f(s,x) = \phi(s,x,u(s)),$$

where ϕ is a normal integrand on $S \times R^n \times R^k$ and $u\colon S \to R^k$ is measurable. Then f is normal.

PROOF. Apply the second assertion of Proposition 2N with $g(s,x) = (x,u(s))$. Q.E.D.

2Q. COROLLARY. *Let f be an integrand on $S \times R^n$ of the form*

$$f(s,x) = \lambda(s)g(s,x),$$

where g is a proper normal integrand on $S \times R^n$, $\lambda\colon S \to R_+$ is measurable, and either of the conventions $0\cdot\infty = 0$ or $0\cdot\infty = \infty$ is used. Then f is normal.

PROOF. Apply the first assertion of Proposition 2N with $\phi(s,\alpha) = \lambda(s)\alpha$; this yields the result for $0\cdot\infty = \infty$. The case of $0\cdot\infty = 0$ is then obtained simply by redefining $f(s,\cdot)$ to be identically 0 on the (measurable) set where $\lambda(s) = 0$. Q.E.D.

2R. PROPOSITION. *Let f be an integrand on $S \times R^n$ of the form*

$$f(s,x) = \inf_{u \in R^k} \phi(s,x,u), \tag{2.7}$$

where ϕ is a normal integrand on $S \times R^n \times R^k$. If $f(s,x)$ is lower semicontinuous in x, then f is normal.

(The following growth condition on ϕ is sufficient for $f(s,x)$ to be lower semicontinuous in x, and for the minimum in (2.7) to be attained: for every $s \in S$, every $\alpha \in R$ and every bounded set $K \subset R^n$, the set

$$\{u \in R^k \mid \exists x \in C \text{ with } \phi(s,x,u) \leq \alpha\}$$

is bounded.)

More generally, if f fails to be lower semicontinuous, the integrand

(2.8) $$\bar{f}(s,x) = \liminf_{x' \to x} f(s,x')$$

is nevertheless normal.

PROOF. For the projection $A: (x,u,\alpha) \to (x,\alpha)$, we have $E_{\bar{f}}(s) = \text{cl}\, A(E_\phi(s))$. The normality of $\bar{f}$ is thereby seen to be a consequence of Theorem 1N. If $f(s,x)$ is lower semicontinuous in x, we of course have $f = \bar{f}$. The condition for lower semicontinuity has an elementary proof. Q.E.D.

To conclude this section, we treat some aspects of duality that lead us into convex analysis.

By the conjugate of the integrand f on $S \times R^n$, we shall mean the integrand f^* on $S \times R^n$ defined by

(2.9) $$f^*(s,y) = \sup_{x \in R^n} \{x \cdot y - f(s,x)\}.$$

The biconjugate integrand is given by

(2.10) $$f^{**}(s,x) = \sup_{y \in R^n} \{x \cdot y - f^*(s,y)\}.$$

According to the theory of conjugate convex functions [12], f^* is a closed convex integrand (i.e., $f^*(s,\cdot)$ is for each s a lower semicontinuous convex function, which either does not take on the value $-\infty$ at all or is identically $-\infty$), and f^{**} is the greatest closed convex integrand majorized by f. If f is convex and proper, both f^* and f^{**} are proper.

2S. PROPOSITION. *If f is a normal integrand on $S \times R^n$, then so are the conjugate integrand f^* and the biconjugate integrand f^{**}.*

PROOF. Let $((x_i,\alpha_i) \mid i \in I)$ be a Castaing representation of E_f, and let $T = \text{dom}\, E_f$ (measurable). The Carathéodory integrands

$$g_i(s,y) = x_i(s) \cdot y - \alpha_i(s)$$

on $T \times R^n$ give us the representation

$$f^*(s,y) = \sup_{i \in I} g_i(s,y) \quad \text{for } s \in T,$$

and hence f^* is normal relative to $T \times R^n$. On the other hand, for $s \notin T$ we have $f(s,x) = +\infty$ for all x, and consequently $f^*(s,y) = -\infty$ for all y, i.e., $E_{f^*}(s) = R^{n+1}$. Thus E_{f^*} is measurable relative to T and constant relative to $S \setminus T$. It follows that E_{f^*} is measurable relative to S, and hence f^* is normal. Since f^{**} is the integrand conjugate to f^*, it too must be normal. Q.E.D.

2T. COROLLARY. *Let $\Gamma: S \to R^n$ be a multifunction whose values are closed cones, and let $\Gamma^*(s)$ be the polar of $\Gamma(s)$. If Γ is*

measurable, then so is Γ^*.

PROOF. If $f = \psi_\Gamma$ (cf. (2.4)), then $f^* = \psi_{\Gamma^*}$. Apply 2S and 2H. Q.E.D.

2U. COROLLARY. *Let* $\Gamma: S \to R^n$ *be a closed-convex-valued multifunction. Then* Γ *is measurable if and only if its support function*

(2.11) $$h(s,y) = \sup\{x\cdot y \mid x \in \Gamma(s)\}$$

is a normal (convex) integrand.

PROOF. If $f = \psi_\Gamma$, then $f^* = h$ and $f^{**} = f$. Apply 2S and 2H. Q.E.D.

2V. COROLLARY. *Let* f *be a proper integrand on* $S \times R^n$. *Then* f *is normal and convex if and only if there is a countable collection* $((a_i,\alpha_i) \mid i \in I)$ *comprised of measurable functions* $a_i: S \to R^n$ *and* $\alpha_i: S \to R$, *such that*

$$f_i(s,x) = \sup_{i\in I}\{x\cdot a_i(s) - \alpha_i(s)\}.$$

Similarly, a multifunction $\Gamma: S \to R^n$ *is closed-convex-valued if and only if there is such a collection yielding a representation*

$$\Gamma(s) = \{x \in R^n \mid x\cdot a_i(s) \le \alpha_i(s) \text{ for all } i \in I\}.$$

PROOF. For f, the sufficiency follows from Proposition 2L (the functions in the supremum being Carathéodory integrands), while the necessity is obtained by taking the collection to be any Castaing representation for E_{f^*}. (One has f^* normal and $f^{**} = f$.) For Γ, the sufficiency is justified by Theorem 2J, and the necessity is seen via any Castaing representation of E_h, where h is the normal integrand in Corollary 2U. Q.E.D.

For a convex integrand f on $S \times R^n$, there is associated with each $s \in S$ the *subdifferential multifunction* $\partial f(s,\cdot): R^n \to R^n$, defined by

(2.12) $$\partial f(s,x) = \{y \in R^n \mid f(s,x') \ge f(s,x) + y\cdot(x'-x) \text{ for all } x'\}.$$

This is closed-convex-valued, and its graph is closed, if $f(s,\cdot)$ is lower semicontinuous. If $f = \psi_\Gamma$ (cf. Proposition 2H), the set $\partial f(s,x)$ is the *cone of normals* to $\Gamma(s)$ at x.

The following theorem was first proved by Attouch [14] in a somewhat different infinite-dimensional setting.

2W. THEOREM. *Let* f *be a lower semicontinuous proper convex integrand on* $S \times R^n$. *Then the following are equivalent*:

(a) f *is a normal integrand;*

(b) *(Attouch's condition): the graph of the closed-valued*

multifunction $\partial f(s,\cdot)$ *depends measurably on* s, *and there is at least one measurable function* $x\colon S \to R^n$ *such that* $f(s,x(s))$ *is finite and measurable in* s *and* $\partial f(s,x(s)) \neq \emptyset$ *for all* $s \in S$.

PROOF. (a) $\Rightarrow$ (b). Let

$$g(s,x,y) = f(s,x) + f^*(s,y) - x\cdot y,$$

so that

$$\operatorname{gph} f(s,\cdot) = \{(x,y) \mid g(s,x,y) \leq 0\}.$$

In view of Propositions 2S and 2M, g is a normal integrand, and this representation therefore shows that $\operatorname{gph} f(s,\cdot)$ depends measurably on s (Proposition 2I). Furthermore, this graph is nonempty for every s, because $f(s,\cdot)$ is a proper convex function [12,p.217]. Hence there exist by Corollary 1C measurable functions $x\colon S \to R^n$ and $y\colon S \to R^n$, such that $y(s) \in \partial f(s,x(s))$ for every s. This implies $f(s,x(s))$ is finite; of course, $f(s,x(s))$ is measurable in s by Corollary 2B.

(b) $\Rightarrow$ (a). Let $((x_i,y_i) \mid i \in I)$ be a Castaing representation of the multifunction $\Gamma(s) = \operatorname{gph} f(s,\cdot)$; this can be chosen so that, for a certain index i_0, $f(s,x_{i_0}(s))$ is finite and measurable in s. It is known from [12, Theorem 24.9 and proof of Theorem 24.8] that $f(s,x)$ is the supremum of

$$f(s,x_0(s)) + (x_{i_1}(s)-x_{i_0}(s))\cdot y_{i_0}(s)+(x_{i_2}(s)-x_{i_1}(s))\cdot y_{i_1}(s)$$

$$+\cdots+(x-x_{i_m}(s))\cdot y_{i_m}(s)$$

over all finite families $(i_k \mid k=1,\ldots,m)$ of indices in I. Each of the expressions in the supremum, viewed as a function of (s,x), is a Carathéodory integrand. Thus f is the supremum of a countable family of Carathéodory integrands, and the normality of f follows from Proposition 2L. Q.E.D.

2X. COROLLARY. *Let* f *be a normal proper convex integrand on* $S \times R^n$, *and let*

$$\Gamma(s) = \partial f(s,x(s)),$$

where $x\colon S \to R^n$ *is measurable. Then* Γ *is measurable (closed-valued).*

PROOF. In view of 2W, this is a special case of Theorem 1N. Q.E.D.

3. Integral Functionals on Decomposable Spaces.

From now on, we denote by μ a nonnegative, σ-finite measure on $(S,\mathcal{A})$.

For any normal integrand f on $S \times R^n$ and any measurable function $x\colon S \to R^n$, we have $f(s,x(s))$ measurable in s, and therefore the integral

$$I_f(x) = \int_S f(s,x(s))\mu(ds)$$

has a well defined value in $\overline{R}$ under the following convention: if neither the positive nor the negative part of the function $s \to f(s,x(s))$ is summable (i.e., finitely), we set $I_f(x) = +\infty$. In particular, then,

$$I_f(x) < +\infty \Rightarrow f(s,x(s)) < +\infty \text{ a.e.} \tag{3.1}$$

We call I_f the integral functional associated with the integrand f. Typically, we are concerned with the restriction of I_f to some linear space X of measurable functions $x\colon S \to R^n$. Notice that I_f is a convex functional on X, if f is a normal convex integrand.

Among the linear spaces X of interest, besides the space of all measurable functions, are the various Lebesgue spaces and Orlicz spaces, the space of constant functions, and in the case of topological or differentiable structure on S, spaces of continuous or differentiable functions. In their role in the theory of integral functionals, however, these spaces fall into two very different categories, distinguished by the presence or absence of a certain property of decomposability.

Slightly generalizing the original definition in [1], we shall say that X, a linear space of measurable functions $x\colon S \to R^n$, is decomposable if S can be expressed as the union of an increasing sequence of measurable subsets S_k $(k=1,2,\ldots)$, such that for every S_k and bounded measurable function $x'\colon S_k \to R^n$, and every $x'' \in X$, the (measurable) function

$$x(s) = \begin{cases} x'(s) & \text{for } s \in S_k, \\ x''(s) & \text{for } s \in S\backslash S_k, \end{cases} \tag{3.2}$$

belongs to X. (The original definition required this property, not just for S_k, but all measurable sets $T \subset S$ with $\mu(T)$ finite.) Since μ is σ-finite, the sets S_k can always be chosen with $\mu(S_k)$ finite.

The space of all measurable functions, the Lebesgue spaces and Orlicz spaces, are all decomposable. However, the space of constant functions and spaces of continuous or differentiable functions furnish examples of nondecomposability.

The concept of decomposability is designed for the following result.

3A. THEOREM. *Let* f *be a normal integrand on* $S \times R^n$, *and let* X *be a linear space of measurable functions* $x: S \to R^n$. *For the relation*

$$\inf_{x \in X} \int_S f(s,x(s))\mu(ds) = \int_S [\inf_{x \in R^n} f(s,x)]\mu(ds) \tag{3.3}$$

to hold, it is sufficient that X *be decomposable and that the first infimum not be* $+\infty$. (*These conditions are superfluous in the case where* X *is the space of all measurable functions, or more generally, if* f *satisfies a condition implying that* X *contains every measurable function* x *such that* $I_f(x) < +\infty$.)

PROOF. The expression integrated on the right side of (3.3) is

$$m(s) = \inf_{x \in R^n} f(s,x),$$

which is measurable by Theorem 2K; as in the definition of I_f, this integral is considered to be $+\infty$ if neither the positive nor the negative part of m is summable. For each measurable function x, we have $f(s,x(s)) \geq m(s)$ for all $s \in S$. Thus the inequality $\geq$ is trivial in (3.3), and our task is to show, assuming

$$\int_S m(s)\mu(ds) < \beta < +\infty,$$

that there exists $x \in X$ satisfying $I_f(x) < \beta$. Since μ is σ-finite, there is a *positive* function $p: S \to R$ such that $\int_S p(s)\mu(ds) < \infty$. Setting

$$\alpha(s) = \varepsilon p(s) + \max\{m(s), -\varepsilon^{-1}\}$$

for $\varepsilon > 0$ sufficiently small, we have a measurable function $\alpha: S \to R$ such that $\alpha(s) > m(s)$ for all s, and $\int_S \alpha(s)\mu(ds) < \beta$. The multifunction

$$\Gamma(s) = \{x \in R^n | f(s,x) \leq \alpha(s)\}$$

is then nonempty-closed-valued and, by Proposition 2I, measurable. Hence there is a measurable function $x': S \to R^n$ such that $f(s,x'(s)) \leq \alpha(s)$ for all s (Corollary 1C) and consequently $I_f(x') < \beta$. However, x' need not belong to X (except in the cases covered by the parenthetical remarks in the theorem), so in general a modification of x' is needed. Let $x'' \in X$ be such that $I_f(x'') < +\infty$, and let $(S_k | k=1,2,\ldots)$ be as in the definition of decomposability. Intersecting each S_k with the measurable set $\{s \in S| \, |x'(s)| \leq k\}$ if necessary, we can suppose x' to be bounded on S_k. Since $I_f(x') < \beta$ and $I_f(x'') < +\infty$, we have for all k sufficiently large that

$$\int_{S_k} f(s,x'(s))\mu(ds) + \int_{S \setminus S_k} f(s,x''(s)) < \beta.$$

Thus for x defined as in (3.2), we have $I_f(x) < \beta$, and by our decomposibility assumption, $x \in X$. Q.E.D.

As an important illustration of how Theorem 3A can be applied, let us consider an optimization problem of the form:

(Q) $\quad$ minimize $J(x) + I_\phi(x,u)$ over all $x \in X$, $u \in U$,

where ϕ is a normal integrand on $S \times R^n \times R^k$, X and U are linear spaces of measurable functions $x: S \to R^n$ and $u: S \to R^k$, and the functional $J: X \to \overline{R}$ is arbitrary. (To cover all contingencies, we adopt the convention $\infty - \infty = +\infty$ in (Q).) The question to be investigated is whether (Q) is equivalent to the reduced problem

(P) $\quad$ minimize $J(x) + I_f(x)$ over all $x \in X$,

where

$$f(s,x) = \inf_{u \in R^k} \phi(s,x,u). \tag{3.4}$$

Here f is normal by Proposition 2R if, as we now assume, $f(s,x)$ is lower semicontinuous in x (cf. the sufficient condition for lower semicontinuity furnished in 2R).

3B. COROLLARY. (*Theorem on Reduced Minimization*). *In the above context of problems (Q) and (P), suppose further that*

(i) the infimum defining $f(s,x)$ in (3.4) is always attained (cf. the sufficient condition given in Proposition 2R), and

(ii) whenever $u: S \to R^k$ is a measurable function yielding $J(x) + I_\phi(x,u) < +\infty$ for some $x \in X$, one necessarily has $u \in U$.

Then (P) and (Q) are equivalent, in the sense that for every $x \in X$ with $J(x) < +\infty$, one has

$$I_f(x) = \inf_{u \in U} I_\phi(x,u), \tag{3.5}$$

this infimum always being attained by at least one $u \in U$.

PROOF. Fix any $x \in X$ with $J(x) < +\infty$, and define $g(s,u) = \phi(s,x(s),u)$. Then g is a normal integrand (Corollary 2P), and it follows from Theorem 2A and assumption (ii) that

$$\inf_{u \in U} I_g(u) = \int_S [\inf_{u \in R^k} g(s,u)]\mu(ds) = \int_S f(s,x(s))\mu(ds).$$

Thus (3.5) holds. If the infimum over U is $+\infty$, it is of course attained by every $u \in U$, so let us suppose it is not $+\infty$; then $I_f(x) < +\infty$. The closed-valued multifunction $\Gamma: S \to R^k$ defined by

$$\Gamma(s) = \{s \mid g(s,u) \le f(s,x(s))\}$$

is measurable by Proposition 2I and nonempty-valued by assumption (i). Hence it has a measurable selection u. We have

$$I_\phi(x,u) = I_g(u) \leq I_f(x) < +\infty,$$

which entails $u \in U$ by (ii), and thus u furnishes the minimum in (3.5). Q.E.D.

The wide range of problems where this reduction theorem can be applied is apparent, if it is recalled that very general constraints are representable in terms of the designation of the elements where J and ϕ have the value $+\infty$. The result generalizes, for example, one constituting a key step in establishing the existence of optimal trajectories in control theory; see Rockafellar [15]. It also furnishes, in combination with all the machinery for verifying normality, a powerful tool for the analysis of multistage stochastic optimization problems. Such problems can be reduced to "dynamic programming" more efficiently than has previously been shown, e.g. by Wets and the author [16] and Evstigneev [17].

In the rest of this section, we denote by X and Y two linear spaces of R^n-valued functions such that

$$(3.6) \qquad \int_S |x(s)\cdot y(s)|\mu(ds) < +\infty \quad \text{for all } x \in X,\ y \in Y.$$

The bilinear form

$$\langle x,y\rangle = \int_S x(s)\cdot y(s)\mu(ds)$$

defines a pairing between X and Y, in terms of which the standard theory of locally convex spaces can be applied. In particular, the weak topologies $\sigma(X,Y)$ and $\sigma(Y,X)$ are available. (Strictly speaking, these are not, of course, Hausdorff topologies unless we identify elements of X producing the same linear functional on Y via $\langle\cdot,\cdot\rangle$ and similarly for elements of Y. This identification is harmless, but a potential nuisance for terminology and notation in what follows, so we gloss over it, leaving the details implicit.)

An important case to be borne in mind is that of the (decomposable) Lebesgue spaces: $X = L_n^p$ and $Y = L_n^q$, $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$, where

$$L_n^p = L^p(S,\mathcal{A},\mu;R^n).$$

The relation $(1/p) + (1/q) = 1$ suffices for (3.6), but it is not totally necessary; for instance, it is occasionally useful to employ $p = \infty$ and $q = \infty$ in the case where $\mu(S) < \infty$.

The conjugate on Y of a functional $F\colon X \to \overline{R}$ is, of course,

defined by

$$F^*(y) = \sup_{x \in X}\{\langle x,y\rangle - F(x)\},$$

and similarly the conjugate on X of a functional $G:Y \to \overline{R}$; thus

$$F^{**}(x) = \sup_{y \in Y}\{\langle x,y\rangle - F^*(y)\}.$$

As is well-known, F^* is convex and l.s.c. with respect to $\sigma(Y,X)$; F^{**} is the $\sigma(X,Y)$-l.s.c. convex hull of F, if that functional nowhere has the value $-\infty$, while otherwise $F^{**} \equiv -\infty$.

Our aim now is to apply these facts to integral functionals, making use of Theorem 3A and the normality of the conjugate integrands f^* and f^{**} in Proposition 2S. The next theorem is a slightly improved version of the main result of Rockafellar [1], as extended in [8]. The version in [8] was presented in terms of a separable reflexive Banach space in place of R^n, but with the measurable space (S,A) complete. For a recent generalization, see Valadier [11].

3C. THEOREM. *Let f be a normal integrand on $S \times R^n$, and consider I_f on X. Suppose X is decomposable, and there exists at least one $x \in X$ with $I_f(x) < +\infty$. Then $I_f^* = I_{f^*}$ on Y, and hence in particular the convex functional I_{f^*} on Y is $\sigma(Y,X)$-l.s.c.*

If Y is likewise decomposable, and there exists at least one $y \in Y$ with $I_{f^*}(y) < +\infty$, then $I_f^{**} = I_{f^{**}}$ on X.

PROOF. Fix any $y \in Y$, and consider the integrand

$$g(s,x) = f(s,x) - \langle x,y(s)\rangle.$$

The second term in this expression constitutes a Carathéodory integrand (hence a normal integrand), so g is normal by Proposition 2M. Applying Theorem 3A to g, we obtain

$$\inf_{x \in X} \int_S [f(s,x(s)) - \langle x(s),y(s)\rangle]\mu(ds) = \int[-f^*(s,y(s))]\mu(ds),$$

the common value not being $+\infty$. Due to the latter, it is legitimate to rewrite the equation as

$$\inf_{x \in X}\{I_f(x) - \langle x,y\rangle\} = -I_{f^*}(y),$$

or in other words, $I_f^*(y) = I_{f^*}(y)$. The rest of the theorem follows by duality. Q.E.D.

3D. COROLLARY. *Let f be a normal proper convex integrand on $S \times R^n$. If Y is decomposable and there exists at least one $y \in Y$ with $I_{f^*}(y) < +\infty$, then the convex integral functional I_f on X is $\sigma(X,Y)$-lower semicontinuous (and nowhere $-\infty$).*

PROOF. Apply Theorem 3C to I_{f*} to see that $I^*_{f*} = I_{f**}$. The hypothesis on f is equivalent to the property that $f^{**} = f$. Q.E.D.

3E. COROLLARY. *Let f be a normal convex integrand on $S \times R^n$ such that $I_f(x) < +\infty$ for at least one $x \in X$. Then for every $x \in X$ the subdifferential*

$$\partial I_f(x) = \{y \in Y \mid I_f(x') \geq I_f(x) + \langle x'-x,y\rangle, \quad \forall x' \in X\} \tag{3.7}$$

is given by

$$\partial I_f(x) = \{y \in Y \mid y(s) \in \partial f(s,x(s)) \quad \text{a.e.}\}.$$

PROOF. According to the definition (3.7), we have $y \in \partial I_f(x)$ if and only if

$$\langle x,y\rangle - I_f(x) = I^*_f(y),$$

where $I^*_f(y) = I_{f*}(y)$ by Theorem 3C. The result now follows from the fact that

$$\langle x(s),y(s)\rangle - f(s,x(s)) \leq f^*(s,y(s))$$

always holds, with equality if and only if $y(s) \in \partial f(s,x(s))$. Q.E.D.

3F. COROLLARY. *Let $\Gamma: S \to R^n$ be a measurable, closed-valued multifunction, and let*

$$C = \{x \in X \mid x(s) \in \Gamma(s) \quad \text{a.e.}\},$$

$$h(s,y) = \sup_{x \in \Gamma(s)} x\cdot y \quad \text{for} \quad (s,y) \in S \times R^n.$$

If X is decomposable and $C \neq \emptyset$, then

$$\sup_{x \in C} \langle x,y\rangle = I_h(y) \quad \text{for all} \quad y \in Y.$$

If in addition Y is decomposable, then

$$\sigma(X,Y)\text{-cl co } C = \{x \in X \mid x(s) \in \text{cl co}\Gamma(s) \quad \text{a.e.}\}.$$

PROOF. Let $f = \psi_\Gamma$ (cf. (2.4)); then $f^* = h$, and the result follows at once from Theorem 3C. Q.E.D.

In many situations, it is useful to be able to apply 3C and 3D without very explicit knowledge of the integrand f^*, and this requires some indirect criterion for the existence of $y \in Y$ satisfying $I_{f*}(y) < +\infty$. One case which falls out immediately, is that where there is a lower bound

$$f(s,x) \geq \beta(s) \quad \text{for all} \quad x \in R^n,$$

with β summable; then $f^*(s,0) \leq -\beta(s)$, so $I_{f*}(0) < +\infty$. Another criterion is provided by the next result.

3G. PROPOSITION. *Let f be a normal convex integrand on $S \times R^n$, and let $Y = L^p_n$, $1 \leq p \leq \infty$. Then for the existence of at least one*

$y \in Y$ <u>such that</u> $I_{f*}(y) < +\infty$, <u>the following condition is sufficient: for some</u> $\bar{x} \in L_n^q$ (<u>where</u> $1/p + 1/q = 1$) <u>and some</u> $\varepsilon > 0$, <u>the function</u> $s \to f(s,\bar{x}(s) + u)$ <u>belongs to</u> L_1^p <u>for each</u> $u \in R^n$ <u>satisfying</u> $|u| \leq \varepsilon$, <u>while</u> $I_f(\bar{x}) > -\infty$.

PROOF. Let $\{a_1,\ldots,a_m\} \subset R^n$ be any finite set whose convex hull contains the unit ball; then

$$\max_{i=1}^{m} a_i \cdot y \geq |y| \quad \text{for all} \quad y \in R^n. \tag{3.8}$$

Let $\delta > 0$ be small enough that $|\delta a_i| \leq \varepsilon$ for all i. Then each of the functions

$$\alpha_i(s) = f_i(s,\bar{x}(s) + a_i), \quad i = 1,\ldots,m$$

belongs to L_1^p, as does $\bar{\alpha}(s) = f(s,\bar{x}(s))$. There is a measurable set $T \subset S$ with $\mu(S\backslash T) = 0$, such that these functions are all finite on T. For each $s \in T$, the convex function $f(s,\cdot)$ is finite on a neighborhood of $\bar{x}(s)$ and therefore has $\partial f(s,\bar{x}(s)) \neq \emptyset$. Thus the multifunction $s \to \partial f(s,\bar{x}(s))$ is almost everywhere nonempty-valued; since it is also closed-valued and measurable by 2X, it has a measurable selection relative to the set where it is nonempty-valued (1C). Hence there is a measurable function $\bar{y}: S \to R^n$ satisfying

$$\bar{y}(s) \in \partial f(s,\bar{x}(s)) \quad a.e. \tag{3.9}$$

We then have, almost everywhere,

$$f_i(s,\bar{x}(s) + \delta a_i) \geq f_i(s,\bar{x}(s)) + \delta a_i \cdot \bar{y}(s), \quad i = 1,\ldots,m,$$

or in terms of the notation introduced above,

$$a_i \cdot \bar{y}(s) \leq \delta^{-1}[\alpha_i(s) - \bar{\alpha}(s)], \quad i = 1,\ldots,m.$$

Taking the maximum on both sides with respect to i and recalling (3.8), we obtain $|\bar{y}(s)| \leq \alpha(s)$ a.e., where $\alpha \in L_1^p$. This shows that $\bar{y} \in Y$. Since (3.9) implies

$$f^*(s,\bar{y}(s)) = \langle\bar{x}(s),\bar{y}(s)\rangle - f(s,\bar{x}(s)),$$

while $I_f(\bar{x}) > -\infty$, we have $I_{f*}(\bar{y}) < +\infty$. Q.E.D.

In Theorem 2C, I_{f**} turns out to be the "closed convex hull" of I_f. However, in an important case connected with the theory of "relaxed" variational problems, I_{f**} is also simply the "closure" of I_f; in other words, convexity follows from weak lower semicontinuity. This case is delineated next.

We shall say that the integrand f is <u>atomically convex</u> if, for each atom $T \subset S$, *the function* $f(s,\cdot)$ is convex for almost every $s \in T$. Of course, if the measure space $(S,\mathcal{A},\mu)$ is without atoms, this condition is automatically satisfied.

3H. THEOREM. Let f be a normal integrand on $S \times R^n$ which is atomically convex. Suppose X and Y are both decomposable and contain elements x and y such that $I_f(x) < +\infty$ and $I_{f*}(y) < +\infty$.

Then the proper convex functional I_{f**} is the greatest $\sigma(X,Y)$-l.s.c. functional on X majorized by I_f. In fact, I_f itself is $\sigma(X,Y)$-l.s.c. if and only if $f(s,x)$ is convex in x for almost every s.

PROOF. To prove the first assertion, it is enough, in view of Theorem 3C, to demonstrate that the weak closure of the (nonempty) set

$$\text{epi } I_f = \{(x,\alpha) \in X \times R \mid \alpha \geq I_f(x)\}$$

is convex. Remembering the nature of the topology $\sigma(X,Y)$, one sees this is equivalent to showing that the closure of the image of epi I_f under any mapping of the form

$$(x,\alpha) \to (\langle x,y_1\rangle + \alpha\beta_1, \ldots, \langle x,y_m\rangle + \alpha\beta_m)$$

is convex. Here we have $\alpha = \int_S \lambda(s)\mu(ds)$ for some $\lambda \in L_1^1$ such that $\lambda(s) \geq f(s,x(s))$ a.e., so the question can be rephrased as follows. Let $Z = X \times L_1^1$ (this being a decomposable space of R^{n+1}-valued functions, since X is decomposable), and let

$$C = \{z \in Z \mid z(s) \in E_f(s) = \text{epi } f(s,\cdot) \ \text{ a.e.}\}$$

(a nonempty set because epi I_f is nonempty). Consider any linear transformation $A\colon Z \to R^m$ of the form

$$Az = \int_S M(s)z(s)\mu(ds), \qquad z \in Z,$$

where $M(s)$ is a matrix of dimension $m \times (n+1)$ whose components are such that $M(s)z(s)$ is summable for every $z \in Z$. It suffices to show that cl(AC) is convex. Passing to $C - \bar{z}$ if necessary, where $\bar{z}$ is any particular element of C, it can be supposed in this that $0 \in C$.

Let $(S_k \mid k = 1,2,\ldots)$ be a family of measurable sets with the property in the definition of decomposability, and for each $r > 0$ and measurable set T which is contained in S_k for all k sufficiently large, let C_T^r denote the set of all measurable functions $z\colon T \to R^{n+1}$ satisfying

(3.10) $\qquad z(s) \in E_f(s)$ and $|z(s)| \leq r$ for all $s \in T$.

The decomposability property implies C_T^r is the same as the set of all restrictions to T of functions $z \in Z$ satisfying (3.10), and in fact (since $0 \in C$) any $z \in C_T^r$ can be extended to an element of C by giving it the zero value outside of T. Thus for the mapping

$$A_T z = \int_T M(s)z(s)\mu(ds)$$

we have $AC \supset A_T C_T^r$, where the latter set increases with T and r. For any $z \in C$ and $\varepsilon > 0$, the set

$$T = S_k \cap \{s \mid z(s) \in E_f(s) \text{ and } |z(s)| \leq r\}$$

yields $|Az - A_T z| < \varepsilon$ for k and r sufficiently large, and one has $A_T z \in A_T C_T^r$. Therefore

$$\text{cl } AC = \text{cl} \cup A_T C_T^r,$$

where $A_T C_T^r$ increases with r and T; the union is respect to all $r > 0$ and measurable T such that $T \subset S_k$ for k sufficiently large. The problem can therefore be reduced to showing that each of the sets of the form $A_T C_T^r$ is convex. (For this purpose, we note that the components of $M(s)$ in the definition of A_T must actually be summable over $s \in T$, since $M(s)z(s)$ is by assumption summable over T for every $z \in Z$, and by the decomposability property the set of restrictions to T of the functions in Z includes all bounded measurable functions.)

The convexity of $A_T C_T^r$ will be shown to follow from the well-known theorem of Liapunov, which asserts that the range of a nonatomic R^n-valued measure is convex, in fact compact. (For a short proof of Liapunov's theorem using the Krein-Milman Theorem, see Lindenstrauss [18]; the Hahn decomposition theorem can be used to remove the assumption of Lindenstrauss that the component measures are nonnegative.) First we partition S into S_0 and S_1, where μ is purely atomic relative to S_0 and nonatomic relative to S_1. Let $T_0 = T \cap S_0$ and $T_1 = T \cap S_1$. According to our hypothesis that f is atomically convex, we have $E_f(s)$ convex for almost every $s \in T_0$, and hence $C_{T_0}^r$ is convex. Since

$$A_T C_T^r = A_{T_0} C_{T_0}^r + A_{T_1} C_{T_1}^r,$$

convexity of $A_T C_T^r$ will follow from that of $A_{T_1} C_{T_1}^r$. Let z and z' be any two elements of $C_{T_1}^r$, and define the set function τ, for measurable sets $E \subset T_1$, by

$$\tau(E) = A_E(z' - z) = A_{T_1} z_E,$$

where $z_E(s) = z'(s) - z(s)$ for $s \in E$, and $z_E(s) = 0$ for $s \in T_1 \backslash E$. Obviously τ is countably additive (since the matrix components defining A_{T_1} are, as seen above, summable over T_1), and

$$\tau(E) + A_{T_1} z = A_{T_1}(z_E + z), \text{ with } z_E + z \in C_{T_1}^r.$$

Let $D = (\text{range } \tau) + A_{T_1} z$. Then D is a subset of $A_{T_1} C^r_{T_1}$ which, by Liapunov's theorem, is convex. Moreover, D contains both z (corresponding to $E = \emptyset$) and z' (corresponding to $E = T_1$). The line segment joining z and z' is therefore contained in D, hence in $A_{T_1} C^r_{T_1}$; this shows the latter set is convex.

It remains to demonstrate the final assertion of the theorem. The sufficiency of the condition is covered by 3D (with a slight maneuver around a set of measure zero), so we direct ourselves to the necessity. In view of what has already been proved, our starting assumption is that $I_{f^{**}}(x) = I_f(x)$ for every $x \in X$. Since $f^{**} \leq f$, this implies

(3.11) $$f^{**}(s,x(s)) = f(s,x(s)) \text{ a.e. for each } x \in X.$$

Making use of decomposability, we can express S as the union of an increasing sequence of sets S_k, such that

(3.12) $$f^{**}(s,x(s)) = f(s,x(s)) \text{ for almost every } s \in S_k, \text{ whenever } x\colon S_k \to R^n \text{ is measurable and bounded.}$$

Fix any k and $r > 0$, and consider the (measurable) multifunction Γ defined by

$$\Gamma(s) = E_{f^{**}}(s) \cap [rB \times R],$$

where B is the closed unit ball in R^n. Let $((x_i,\alpha_i) \mid i \in I)$ be a Castaing representation for Γ. Then by (3.12)

$$\alpha_i(s) \geq f^{**}(s,x_i(s)) = f(s,x_i(s))$$

for almost every $s \in S_k \cap \text{dom}\Gamma$, so that (since I is countable) the relation

(3.13) $$(x_i(s),\alpha_i(s)) \in E_f(s) \cap [rB \times R] \text{ for all } i \in I$$

holds for almost every $s \in S_k \cap \text{dom}\Gamma$. Of course, (3.13) implies

$$\Gamma(s) \subset E_f(s) \cap [rB \times R],$$

or what is the same thing,

$$f^{**}(s,x) = f(s,x) \text{ for all } x \in R^n \text{ with } |x| \leq r.$$

This equation has been shown to hold for almost every $s \in S_k$ such that $\Gamma(s) \neq \emptyset$ (i.e. $f^{**}(s,x) < +\infty$ for at least one x with $|x| \leq r$), and it holds trivially if $\Gamma(s) = \emptyset$ (both $f^{**}(s,x)$ and $f(s,x)$ then being $+\infty$). Since k and r are arbitrary, we reach the conclusion that $f^{**}(s,\cdot) = f(s,\cdot)$, except for s in a set of measure zero. Q.E.D.

There are many situations where it is convenient in direct terms to work with integral functionals on the space L_n^∞, because, for example, continuity with respect to the norm is then easier to work with and to express via local properties of the integrand. However, such advantages are often paid for by a troublesome problem when it comes to duality: the dual Banach space $L_n^{\infty *}$ cannot be identified with L_n^1. We shall describe a special result in this direction which shows the situation is not quite as bad as might be imagined, and which can be used to derive some useful compactness theorems.

A (norm) continuous linear functional z on L_n^∞ is said to be <u>singular</u>, if there is an increasing sequence $(S_k | k = 1,2,\ldots)$ of measurable sets satisfying $S = \cup_{k=1}^\infty S_k$, such that, whenever $x \in L_n^\infty$ is a function vanishing almost everywhere outside of some S_k, one has $z(x) = 0$. The set of these forms a linear space we shall denote by L_n^{sing}. A fundamental fact, equivalent to the Hewitt-Yosida theorem [19], is that under the pairing

(3.12) $\langle x,(y,z)\rangle = \langle x,y\rangle + z(x)$ for $x \in L_n^\infty$, $(y,z) \in L_n^1 \times L_n^{sing}$,

the relation

(3.13) $$L_n^{\infty *} = L_n^1 \otimes L_n^{sing}$$

holds as an isometry (subject to the usual identification of "equivalent" functions in L_n^1 and L_n^∞). For a proof of this result in a much broader context (R^n replaced by an infinite-dimensional space), see Levin [20].

The following theorem is taken from Rockafellar [2].

3I. THEOREM. <u>Let</u> f <u>be a normal integrand on</u> $S \times R^n$, <u>and consider</u> I_f <u>on</u> L_n^∞. <u>Suppose the set</u>

$$F = \{x \in L_n | I_f(x) < +\infty\}$$

<u>is nonempty</u>. <u>Then the conjugate of</u> I_f <u>on</u> $L_n^{\infty *}$ <u>is given in terms of the pairing</u> (3.12) <u>by</u>

(3.14) $$I_f^*(y,z) = I_{f^*}(y) + J_F(z) \quad \underline{\text{for all}} \quad y \in L_n^1,\ z \in L_n^{sing},$$

<u>where</u>

$$J_F(z) = \sup_{x \in F} z(x).$$

PROOF. Using Theorem 3C and the definition of the conjugate functional, we obtain

$$I_f^*(y,z) = \sup_{x \in F}\{\langle x,y\rangle + z(x) - I_f(x)\}$$

$$\leq \sup_{x \in F}\{\langle x,y\rangle - I_f(x)\} + \sup_{x \in F} z(x) = I_{f^*}(y) + J_F(z).$$

Thus $\leq$ holds in (3.14), and the main task is to verify the opposite inequality. In this, we can suppose that $I_f(x) > -\infty$ for all $x \in L_n^\infty$, for otherwise $I_f^*(y,z) \equiv +\infty$. Then $f(s,x(s))$ is summable in s for every $s \in F$.

Fix $y \in L_n^1$, $z \in L_n^{\text{sing}}$, $\beta' < I_{f*}(y)$ and $\beta'' < J_F(z)$. It is enough to show that $\beta' + \beta'' < I_f^*(y,z)$. By virtue of Theorem 3C, we can choose x' and x'' in F such that

$$\beta' < \langle x',y\rangle - I_f(x') = \int_S [\langle x'(s),y(s)\rangle - f(s,x'(s))]\mu(ds)$$

and $\beta'' \leq z(x'')$. Let $(S_k \mid k = 1,2,\ldots)$ be a sequence of sets having the property relative to z that is described in the definition of "singular functional", and define

$$x_k(s) = \begin{cases} x'(s) & \text{for } s \in S_k, \\ x''(s) & \text{for } s \in S \setminus S_k. \end{cases}$$

Then $z(x_k - x'') = 0$, so that $z(x_k) = z(x'') > \beta''$ for all k. On the other hand, because $f(s,x'(s))$ and $f(s,x''(s))$ are both summable over $s \in S$, we have $x_k \in F$ and

$$\langle x_k,y\rangle - I_f(x_k) = \int_{S_k} [\langle x'(s),y(s)\rangle - f(s,x'(s))]\mu(ds)$$

$$+ \int_{S\setminus S_k} [\langle x''(s),y(s)\rangle - f(s,x''(s))]\mu(ds),$$

so that

$$\lim_{k\to\infty}[\langle x_k,y\rangle - I_f(x_k)] = \langle x',y\rangle - I_f(x').$$

Therefore, choosing k sufficiently large, we have

$$\beta' + \beta'' < \langle x_k,y\rangle - I_f(x_k) + z(x_k)$$

$$= \langle x_k,(y,z)\rangle - I_f(x_k) \leq I_f^*(y,z),$$

as desired. Q.E.D.

As a corollary, we state a slight generalization of the main result of [10].

3J. COROLLARY. Let f be a normal integrand on $S \times R^n$, such that I_f, considered as a functional on L_n^1, is not identically $+\infty$, and the set

(3.15) $$G = \{y \in L_n^\infty \mid I_f - \langle\cdot,y\rangle \text{ is bounded below on } L_n^1\}$$
$$= \{y \in L_n^\infty \mid I_{f*}(y) < +\infty\}$$

is nonempty. Let I_f *be the convex functional on* L_n^* *defined in terms of the canonical isomorphism* (3.14) *by*

$$I_f(x,z) = I_{f**}(x) + J_G(z) \text{ for } x \in L_n^1,\ z \in L_n^{\mathrm{sing}},$$

where

$$J_G(z) = \sup_{y \in G} z(y).$$

Then I_f *is the greatest* $\sigma(L_n^{\infty*}, L_n^\infty)$*-l.s.c. convex functional on* $L_n^{\infty*}$ *majorized by* I_f *on* L_n^1 *(regarded as a subspace of* $L_n^{\infty*}$*).*

In fact, if f *is atomically convex,* I_f *is simply the greatest* $\sigma(L_n^{\infty*}, L_n^\infty)$*-l.s.c. functional on* $L_n^{\infty*}$ *majorized by* I_f *on* L_n^1*. Then for each*

$$\alpha > \inf\{I_f(x) \mid x \in L_n^1\},$$

one has

$$\{(x,z) \mid I_f(x,z) \le \alpha\} = \sigma(L_n^{\infty*}, L_n^\infty)\text{-cl}\{x \mid I_f(x) \le \alpha\}.$$

PROOF. We have $I_f^* = I_{f*}$ by Theorem 3C (justifying the equivalence of the two expressions for G in (3.15)). Applying Theorem 3I to I_f, we get $I_f = I_{f*}^* = I_f^{**}$ (with respect to the extended pairing), and this yields the first result. The second result is then immediate from Theorem 3H. Q.E.D.

A functional $F: X \to \bar{R}$ is said to be $\sigma(X,Y)$-*inf-compact* if all its level sets of the form $\{x \in X \mid F(x) \le \alpha\}$, $\alpha \in R$, are $\sigma(X,Y)$-compact. It is $\sigma(X,Y)$-*coercive* if $F - \langle\cdot, y\rangle$ has this property for every $y \in Y$.

Our next objective is to state a rather complete criterion for these properties, in the case of an integral functional I_f on L_n^p, and their duality with continuity properties of I_{f*}. It will be seen that many properties, which might in general be expected to be distinct, collapse into equivalence when the measure space is without atoms.

The following growth conditions on an integrand f on $S \times R^n$ will be crucial:

(G_1): For each $r \ge 0$, there exists $b \in L_1^1$ such that, for almost every $s \in S$,

$$f(s,x) \ge r|x| - b(s) \text{ for all } x \in R^n.$$

(G_p)($1 < p < \infty$): There exist $r \ge 0$ and $b \in L_1^1$ such that, for almost every $s \in S$,

$$f(s,x) \ge r|x|^p - b(s) \text{ for all } x \in R^n.$$

(G_∞): There exist $r \geq 0$ and $b \in L_1^1$ such that, for almost every $s \in S$,

$$f(s,x) < +\infty \Rightarrow |x| \leq r \text{ and } f(s,x) \geq -b(s).$$

$(G_p^*)(1 \leq p < \infty)$: There exist $a \geq 0$ and $b \in L_1^1$ such that, for almost every $s \in S$,

$$f(s,x) \leq a|x|^p + b(s) \text{ for all } x \in R^n.$$

(G_∞^*): $f(s,x)$ is summable in s for all $x \in R^n$.

3K. THEOREM (*Weak Compactness*). *Let f be a normal convex integrand on $S \times R^n$, and let $1 \leq p \leq \infty$, $(1/p) + (1/q) = 1$. Consider I_f on $X = L_n^p$ and I_{f^*} on $Y = L_n^q$. Then among the following conditions the implications (a) $\Leftarrow$ (b) $\Leftarrow$ (c) $\Leftarrow$ (d) $\Leftrightarrow$ (e) are always valid, with the conditions all actually equivalent if the measure space has no atoms:*

(a) *I_f is $\sigma(L_n^p, L_n^q)$-inf-compact and proper on L_n^p;*

(b) *I_f is $\sigma(L_n^p, L_n^q)$-coercive and proper on L_n^p;*

(c) *$I_{f^*}(y)$ is finite for every $y \in L_n^q$;*

(d) *f satisfies the growth condition (G_p), and $I_f(x) < +\infty$ for at least one $x \in L_n^p$;*

(e) *f^* satisfies the growth condition (G_q), and $I_{f^*}(y) > -\infty$ for at least one $y \in L_n^q$;*

REMARK. The convexity of $f(s,x)$ in x, at least for almost every $s \in S$, is necessary for (a) to hold in the case of an atomless measure space. This follows from Theorem 3H.

PROOF. (b) $\Rightarrow$ (a). Trivial.

(c) $\Rightarrow$ (b). In particular, for any finite subset $\{y_1, \ldots, y_m\}$ of L_n^q, the function

$$\alpha(s) = \max_{i=1}^m f^*(s, y_i(s))$$

is summable, and we have

$$f^*(s,y) \leq \alpha(s) \text{ when } y \in \mathrm{co}\{y_1(s), \ldots, y_m(s)\}.$$

This shows that, for almost every s, $f^*(s,\cdot)$ is finite on $\mathrm{co}\{y_1(s), \ldots, y_m(s)\}$. Arguing in this way with various choices of the functions y_i, it is easy to see that, for almost every $s \in S$, $f^*(s,y)$ must be finite for all $y \in R^n$.

Proceeding after this preliminary, we show I_f is proper. Fix any $\bar{y} \in L_n^q$ and let $\Gamma(s) = \partial f^*(s, \bar{y}(s))$. Then Γ is a measurable, closed-valued multifunction (Corollary 2X) and by the finiteness just established, $\Gamma(s) \neq \emptyset$ a.e. Hence Γ has a measurable selection by

Corollary 1C: there exists $\bar{x}: S \to R^n$ such that $\bar{x}(s) \in \partial f^*(s,\bar{y}(s)$ a.e. Then

$$\bar{x}(s)\cdot u(s) \leq f^*(s,\bar{y}(s)+u(s)) - f^*(s,\bar{y}(s)) \text{ for all } u \in L_n^q.$$

In other words, for every $u \in L_n^q$, $\bar{x}\cdot u$ is majorized by a summable function. Therefore $\bar{x} \in L_n^p$. Since $\bar{x}(s) \in \partial f^*(s,\bar{y}(s))$ a.e., we also have

$$f(s,\bar{x}(s)) = \bar{x}(s)\cdot\bar{y}(s) - f^*(s,\bar{y}(s)) \quad \text{(summable)},$$

and hence $I_f(\bar{x}) < +\infty$. Of course, it is trivially true that

$$I_f(x) \geq \langle x,\bar{y}\rangle - I_{f^*}(\bar{y}) \quad \text{for all } x \in L_n^p,$$

and hence $I_f(x) > -\infty$ for all $x \in L_n^p$. Therefore I_f is proper, as claimed.

Since I_f is proper, it follows by Theorem 3C that I_{f^*} is $\sigma(L_n^q,L_n^p)$-l.s.c. and in particular l.s.c. in the norm topology. But a finite convex functional having this property on a Banach space is necessarily continuous everywhere [21; 7C], and its conjugate on the dual Banach space is then weak*-coercive [22], [21]. For $1 \leq q < \infty$, we have $L_n^{q*} = L_n^p$, and $I_{f^*}^* = I_f$ (Theorem 3C), so (b) follows without further ado. For $q = \infty$, the dual Banach space can be identified with $L_n^1 \times L_n^{sing}$ as in Theorem 3I, yielding for the conjugate functional the representation

$$I_{f^*}^*(x,z) = I_f(x) + J_G(z), \tag{3.16}$$

where

$$J_G(z) = \sup\{z(y) \mid y \in L_n^q \text{ with } I_{f^*}(y) < +\infty\}.$$

In fact, $J_G(z) = +\infty$ for all $z \neq 0$, because I_{f^*} is being assumed finite throughout L_n^q. Thus (3.16) tells us that the level sets of $I_{f^*}^*$ are essentially those of I_f, and the weak*-coercivity of $I_{f^*}^*$ is nothing other than the $\sigma(L_n^1,L_n^\infty)$-coercivity of I_f.

(e) $\Rightarrow$ (c). For the case where $1 \leq q < \infty$, we have

$$I_{f^*}(y) \leq a\|y\|^q + \int b\,d\mu < +\infty \quad \text{for all } y \in L_n^q.$$

Since I_{f^*} is a convex functional, this implies either I_{f^*} is finite throughout L_n^q, or $I_{f^*} \equiv -\infty$; but the second possibility has been excluded by assumption. If $q = +\infty$, we again get the finiteness of I_{f^*} (and thereby the same conclusion), in observing the following. Given any $r \geq 0$, choose a finite set $\{y_1,\ldots,y_n\}$ in R^n whose convex hull contains every $y \in R^n$ with $|y| \leq r$. Then, since $f^*(s,\cdot)$ is convex, all such y satisfy

$$f^*(s,y) \leq \max_{i=1}^m f^*(s,y_i) \quad \text{(summable)}.$$

(d) $\Leftrightarrow$ (e). Condition (G_p) is satisfied by f if and only if (G_q^*) is satisfied by f^*, at least for $1 < p \leq \infty (1 \leq q < \infty)$; this is verified by taking conjugates on both sides of the inequalities in question. In the case $p = 1$, $q =$, the exact dual of (G_1) is the assertion that for each $r \geq 0$ there exists $b(s)$ (summable) such that

$$|y| \leq r \Rightarrow f^*(s,y) \leq b(s).$$

This is implied by (G_∞^*), as seen at the end of the preceding paragraph, and it implies in turn that $I_{f^*}(y) < +\infty$ for every $y \in L_n$. The assumption in (d) that $I_f(x) < +\infty$ for some $x \in L_n^p$ yields in all cases

$$(3.17) \qquad I_{f^*}(y) \geq \langle x,y\rangle - I_f(x) > -\infty \quad \text{for all} \quad y \in L_n^q,$$

and combining this with the facts just mentioned we obtain (d) $\Rightarrow$ (e). To complete the verification of (e) $\Rightarrow$ (d), we need only invoke the fact already established, that (e) implies (a) and in particular the properness of I_f.

(a) $\Rightarrow$ (e) <u>for</u> μ <u>nonatomic</u>, $p = 1$, $q = \infty$. We have I_f and I_{f^*} conjugate to each other by Theorem 3C. Therefore, (a) implies I_{f^*} is continuous at 0 in the Mackey topology $\tau = \tau(L_n^\infty, L_n^1)$, and in particular, the convex set

$$G = \{y \in L_n^\infty | \ I_{f^*}(y) < +\infty\}$$

has a nonempty τ-interior containing 0. Therefore, every nonzero linear functional on L_n^∞ which is bounded above on G is τ-continuous and consequently corresponds to an element of L_n^1. If there is no such functional, then, by convexity, G is all of L_n^∞, and we are done (in view of the additional fact that (3.17) holds for any $x \in L_n^1$ with $I_f(x) < +\infty$, and at least one such x is assumed in (a) to exist). Therefore, suppose $0 \neq \bar{x} \in L_n^1$,

$$(3.18) \qquad \infty > \beta \geq \sup_{y \in G} \langle \bar{x}, y\rangle.$$

Let $(y_k | \ k = 1,2,\ldots)$ be a maximizing sequence for the supremum in (3.18). Now define $(y^k | \ k = 0,1,\ldots)$ recursively as follows. To start, $y^o = 0$. Given y^{k-1}, let

$$y^k(s) = \begin{cases} y_k(s) & \text{if} \quad \bar{x}(s)\cdot y_k(s) \geq \bar{x}(s)\cdot y^{k-1}(s), \\ y^{k-1}(s) & \text{if} \quad \bar{x}(s)\cdot y_k(s) < \bar{x}(s)\cdot y^{k-1}(s). \end{cases}$$

Then $y^k \in G$ for all k, and the expression $\bar{x}(s)\cdot y^k(s)$ is nonnegative and nondecreasing in k, with integral bounded above by $\bar{\alpha}$

according to (3.18). Denoting by $\alpha(s)$ the limit as $k \to \infty$, which exists a.e., we have

$$\int \alpha d\mu = \lim_{k\to\infty} \langle \bar{x}, y^k \rangle = \sup_{y\in G} \langle \bar{x}, y \rangle. \tag{3.19}$$

In fact, then

$$y \in G \Rightarrow \bar{x}(s)\cdot y(s) \leq \alpha(s) \quad \text{a.e.,} \tag{3.20}$$

for if y were a function contradicting this implication, we would get a contradiction to $\int \alpha d\mu$ being the supremum in (3.19), by considering, for k sufficiently large, the function $y' \in G$ defined by

$$y'(s) = \begin{cases} y(s) & \text{if } \bar{x}(s)\cdot y(s) \geq \bar{x}(s)\cdot y^k(s), \\ y^k(s) & \text{if } \bar{x}(s)\cdot y(s) < \bar{x}(s)\cdot y^k(s). \end{cases}$$

We thus have

$$G \subset H = \{y \in L_n^\infty \mid \bar{x}(s)\cdot y(s) \leq \alpha(s) \quad \text{a.e.}\}. \tag{3.21}$$

Since G has a nonempty τ-interior, so does H, and it follows that the polar set H^o in L_n^1 is $\sigma(L_n^1, L_n)$-compact. Applying Corollary 3F with

$$\Gamma(s) = \{y \in R^n \mid \bar{x}(s)\cdot y \leq \alpha(s)\},$$

one finds that

$$\sup_{y\in H} \langle x, y \rangle = \int_S \lambda(s)\alpha(s)\mu(ds) \quad \text{if} \quad x(s) = \lambda(s)\bar{x}(s) \quad \text{with } \lambda(s) \geq 0 \text{ a.e.,}$$

and otherwise the supremum is $+\infty$. Thus H^o consists of all measurable functions x of the latter form with

$$\int_S \lambda(s)|\bar{x}(s)|\mu(ds) < \infty \quad \text{and} \quad \int_S \lambda(s)\alpha(s)\mu(ds) \leq 1$$

(where $\alpha(s) \geq 0$). Actually, since G is a τ-neighborhood of 0 in (3.21), it is in particular a neighborhood of 0 in the norm topology, and there exists, therefore, some $\varepsilon > 0$ such that $\varepsilon|\bar{x}(s)| \leq \alpha(s)$ a.e. Hence

$$\int_S \lambda(s)\alpha(s)\mu(ds) \leq 1 \Rightarrow \int_S \lambda(s)|\bar{x}(s)|\mu(ds) < \varepsilon^{-1},$$

and we see that

$$H^o = \{\lambda\bar{x} \mid \lambda(s) \geq 0 \text{ a.e.}, \int \lambda\alpha d\mu \leq 1\}.$$

We claim the $\sigma(L_n^1, L_n^\infty)$-compactness of this set is impossible with μ nonatomic. Indeed, if μ is of this nature, we can find a measurable set T with $0 < \mu(T) < \infty$, together with number $\delta > 0$, such that

$$\delta \leq |\bar{x}(s)| \leq \delta^{-1} \quad \text{and} \quad \delta \leq \alpha(s) \leq \delta^{-1} \quad \text{for all } s \in T.$$

The mapping $\lambda \to \lambda\bar{x}$ is then an isomorphism between the space $L^1(T,\alpha\mu)$ (which is necessarily infinite-dimensional) and a certain subspace of L^1_n, with the property that the image of B_+, the nonnegative part of the unit ball of $L^1(T,\alpha\mu)$, is relatively $\sigma(L^1_n,L^\infty_n)$-compact. This implies, inadmissibly, that B_+ is weakly compact in $L^1(T,\alpha\mu)$ itself, so $L^1(T,\alpha\mu)$ is finite-dimensional.

(a) $\Rightarrow$ (e) <u>for</u> μ <u>nonatomic</u>, $1 < p \leq \infty$, $1 \leq q < \infty$. Again, we have I_f and I_{f*} conjugate to each other by Theorem 3C, and (a) therefore implies I_{f*} is continuous at 0 in the norm topology [21], [22]. In particular, for some $\varepsilon > 0$ and $\beta \in R$, we have

(3.22) $$\|y\| \leq \varepsilon \Rightarrow I_{f*}(y) \leq \beta.$$

Because μ is nonatomic, the maneuver at the beginning of the proof of (c) $\Rightarrow$ (b) above shows (even if the elements y_i in that argument are required to satisfy $\|y_i\| \leq \varepsilon$) that $f_i^*(s,y)$ is finite in y for almost every $s \in S$. For this reason, we can suppose, without loss of generality in the rest of the proof, that actually

(3.23) $f^*(s,y)$ is finite for all $s \in S$ and $y \in R^n$.

Define

(3.24) $$\theta(s,\eta) = \inf\{-f^*(s,y) \mid (|y|/\varepsilon)^q \leq \eta\} \quad \text{for } (s,\eta) \in S \times R,$$

so that

(3.25) $$\theta(s,\eta) \leq -f^*(s,0) \quad \text{if } \eta \geq 0,$$

(3.26) $$\theta(s,\eta) = +\infty \quad \text{if } \eta < 0$$

It will be enough to show the existence of $c \in L^\infty_1$ and $b \in L^1_1$ such that

(3.27) $$\theta(s,\eta) \geq c(s)\eta - b(s) \text{ a.e.},$$

since then by the definition (3.24) of θ we will have

$$f^*(s,y) \leq |c(s)|\eta + b(s) \quad \text{whenever } (|y|/\varepsilon)^q \leq \eta,$$

and consequently

$$f^*(s,y) \leq a|y|^q + b(s) \quad \text{for } a = \|c\|_\infty/\varepsilon^q.$$

We shall obtain this existence by applying some of the preceding theory of integral functionals to I_θ on L^1_1.

To see the normality of θ, we look at the representation

(3.28) $$\theta(s,\eta) = \inf_{y \in R^n} \phi(s,\eta,y),$$

where

$$\phi(s,\eta,y) = \begin{cases} -f^*(s,y) & \text{if } (|y|/\varepsilon)^q \leq \eta, \\ +\infty & \text{otherwise.} \end{cases}$$

We have ϕ itself normal by 2M, because ϕ is the sum of $-f^*$ (normal by 2C) and the indicator of a closed set of pairs of (η,y) that does not depend on s; hence θ is normal by 2R. It is evident from (3.25) and (3.26) that I_θ on L_1^1 has the properties

(3.29) $\qquad I_\theta(\eta) \leq -I_{f^*}(0) < +\infty$ for all $\eta \geq 0$,

(3.30) $\qquad I_\theta(\eta) = +\infty$ for all $\eta \ngeq 0$.

We claim next that

(3.31) $\qquad I_\theta(\eta) \geq -\beta$ for all $\eta \geq 0$ with $\int \eta d\mu \leq 1$.

For, suppose this were violated by a certain $\eta \in L_1^1$. The set

$$\Gamma(s) = \arg\min_{y \in R^n} \phi(s,\eta(s),y)$$

is closed and nonempty by the continuity of $f^*(s,y)$ in y, and Γ is a measurable multifunction by 2K and 2P. Hence by 1C there is a measurable function $y\colon S \to R^n$ such that $y(s) \in \Gamma(s)$ for all s, i.e.

$$-f^*(s,y(s)) = \theta(s,\eta(s)) \quad \text{for all } s,$$

$$(|y(s)|/\varepsilon)^q \leq \eta(s) \quad \text{for all } s.$$

The latter implies $y \in L_n^q$ and $\|y\| \leq \varepsilon$, since $\int \eta d\mu \leq 1$; the former then yields

$$I_{f^*}(y) = -I_\theta(\eta) < \beta,$$

contrary to (3.22). Thus (3.31) holds as claimed.

Now for $k = 1,2,\ldots$ let

(3.32) $\qquad \theta_k(s,\ ) = \max\{\theta(s,\eta),\ \theta(s,0)-k\eta\}.$

Then θ_k is another normal integrand (by 2L), and I_{θ_k} satisfies, like I_θ, the conditions (3.29), (3.30), (3.31). In addition, we have

$$\theta_k^*(s,-k) \leq -\theta(s,0) \quad \text{for all } s \in S,$$

so that, considering $-k$ as a constant function in L_1^∞, we have

$$I_{\theta_k^*}(-k) \leq -I_\theta(0) \leq \beta.$$

The last part of Corollary 3J can therefore be applied to θ_k, the measure μ being nonatomic, and this yields for every

$$\alpha > \inf\{I_{\theta_k^{**}}(\eta) \mid \eta \in L_1^1\}$$

the relation

$$\{\eta \in L_1^1 \mid I_{\theta_k^{**}}(\eta) \leq \alpha\} = \sigma(L_1^1,L_1^\infty)\text{-cl}\{\eta \in L_1^1 \mid I_{\theta_k}(\eta) \leq \alpha\}.$$

In particular, if also $\alpha < -\beta$, the latter implies

$$\{\eta \in L_1^1 \mid I_{\theta_k^{**}}(\eta) \leq \alpha\} \subset \{\eta \in L_1^1 \mid \eta \geq 0, \int \eta d\mu \geq 1\},$$

as follows from (3.30), (3.31) and the weak closedness of the set on the right of (3.35). Thus

$$I_{\theta_k^{**}}(\eta) < -\beta \Rightarrow \eta \geq 0, \int \eta d\mu \geq 1. \tag{3.33}$$

But (3.32) and (3.25) imply

$$-f^*(s,0) \geq \theta_k^{**}(s,\eta) \downarrow \theta^{**}(s,\eta) \quad \text{for} \quad \eta \geq 0,$$

where the first term is summable in s, so that, as ensured by the Lebesgue convergence theorem,

$$I_{\theta^{**}}(\eta) = \lim_{k\to\infty} I_{\theta_k^{**}}(\eta) \quad \text{for all} \quad \eta \geq 0.$$

Therefore by (3.33),

$$I_{\theta^{**}}(\eta) < -\beta \Rightarrow \eta \geq 0, \int \eta d\mu \geq 1.$$

We also have $\theta^{**}(s,\eta) = +\infty$ for $\eta < 0$ by (3.32), and hence

$$I_{\theta^{**}}(\eta) = +\infty \quad \text{if} \quad \eta \not\geq 0.$$

This shows that

$$I_{\theta^{**}}(\eta) \geq -\beta \quad \text{for all} \quad \eta \in L_1^1 \quad \text{with} \quad \|\eta\| \leq 1.$$

Since also by (3.29), we have

$$I_{\theta^{**}}(\eta) < -I_{f^*}(0) < +\infty \quad \text{for all} \quad \eta \geq 0,$$

and we are able to conclude that

$$\liminf_{\|\eta\|\to 0} I_{\theta^{**}}(\eta) \quad \text{is finite.}$$

This implies for the convex functional $I_{\theta^{**}}$ that its conjugate on $L_1^{1*} = L_1^{\infty}$ is proper. But by Theorem 3C this conjugate is

$$I_{\theta^{**}}^* = I_{\theta^{***}} = I_{\theta^*}.$$

Hence there exists at least one $c \in L_1^{\infty}$ with $I_{\theta^*}(c)$ finite. Taking $b(s) = \theta^*(s,c(s))$ (summable), we obtain (3.27) as desired. Q.E.D.

The sufficient condition for compact level sets in L_n^1, given in Theorem 3K by (e) with $q = \infty$, was originally proved in Rockafellar [2], and generalized in [8] to cases with R^n replaced by a Banach space and $(S,\mathcal{A})$ complete. For related results, see also Castaing [23] and Valadier [25]. For versions of the condition which are both sufficient and necessary, see Berliocchi and Lasry [26] and Clauzure [27]. These versions generalize the classical theorem of LaVallée-Poussin.

Theorem 3K and its proof yield, with small effort, the following theorem on continuity. Here the equivalence of (b) and (c) for nonatomic measures reflects facts noted in more general contexts by Bismut [28] and Clauzure [27].

3L. THEOREM (Continuity). Let f be a normal convex integrand on $S \times R^n$, and consider I_f on L_n^p, $1 \leq p \leq \infty$. Then among the following conditions the implications (a) $\Leftrightarrow$ (b) $\Leftarrow$ (c) $\Leftarrow$ (d) always hold, with the conditions all actually equivalent if the measure space is without atoms and $p < \infty$.

(a) I_f is finite on a neighborhood of an element $\bar{x} \in L_n^p$.

(b) I_f is finite and continuous at an element $\bar{x} \in L_n^p$.

(c) I_f is finite and continuous everywhere on L_n^p.

(d) f satisfies the growth condition (G_p^*), and $I_f(x) > -\infty$ for at least one $x \in L_n^p$.

PROOF. Trivially (a) $\Leftarrow$ (b) $\Leftarrow$ (c). The proof of (a) $\Rightarrow$ (b) can be effected by a slight refinement (localization) of the argument in Theorem 3K that (c) $\Rightarrow$ (b). This shows at the same time that (c) is equivalent to the seemingly weaker assertion, (c'), that I_f is finite everywhere on L_n^p. But (d) $\Rightarrow$ (c'), as shown by the beginning of the argument in Theorem 3K that (e) $\Rightarrow$ (c).

(b) $\Rightarrow$ (d) for μ nonatomic, $p < \infty$. Let $g(s,x) = f(s,\bar{x}(s)+x)$. Then g is a normal convex integrand (Proposition 2N). The convex functional I_g is finite and continuous on a neighborhood of the origin in L_n^p, and this implies that the conjugate $I_g^* = I_{g^*}$ by 3C. Applying Theorem 3K to g^*, we see that g satisfies (G_p^*), and hence by the argument just given, I_g is finite and continuous everywhere. Hence I_f is finite and continuous everywhere, and the preceding argument can be retraced with $\bar{x}$ replaced by 0, showing that f itself satisfies (G_p^*). Q.E.D.

3M. PROPOSITION. Let f be a normal convex integrand on $S \times R^n$, and consider I_f on L_n^p, $1 \leq p \leq \infty$. Suppose I_f is finite on a neighborhood of an element $x \in L_n^p$. Then there exists at least one measurable function $y\colon S \to R^n$ satisfying

$$y(s) \in \partial f(s,x(s)) \text{ a.e.,}$$

and moreover every such y belongs to L_n^q $(1/p + 1/q = 1)$ and therefore furnishes an element of $\partial I_f(x)$.

PROOF. This is obtained by reincarnating the proof in Theorem 3K that (c) $\rightarrow$ (b).

For generalizations of Proposition 3M, see Bismut [28, Theorem 2] and Clauzure [27, Prop. 5].

Further properties of integral functionals on L^∞ spaces, connected with the theory of liftings, may be found in Levin [31].

References

1. R.T.Rockafellar, "Integrals which are convex functionals", Pacific J.Math. 24 (1968), 525-539.
2. R.T.Rockafellar, "Integrals which are convex functionals, II," Pacific J. Math. 39 (1971), 439-469.
3. C.Castaing, "Sur les multi-applications mesurables", thèse, Caën, 1967. This has partly been published in Rev. Franc. Info. Rech. Operationelle 1 (1967), 3-34.
4. K. Kuratowski and C. Ryll-Nardzewski, "A general theorem on selectors," Bull. Polish Acad. Sci. 13 (1965), 397-411.
5. V.A.Rokhlin, "Selected topics from the metric theory of dynamical systems", Uspekhi Mat. Nauk 4 (1949), 57-128. See also Amer.Math. Soc.Translations 49 (1966), 171-240.
6. R.T.Rockafellar, "Measurable dependence of convex sets and functions on parameters", J.Math.Anal.Appl. 28 (1969), 4-25.
7. M.F.Sainte-Beuve, "Sur la généralization d'un théorème de section mesurable de von Neumann-Aumann", J.Math.Anal.Appl. 17 (1974), 112-129.
8. R.T.Rockafellar, "Convex integral functionals and duality", Contributions to Nonlinear Functional Analysis (E. Zarantonello, editor), Academic Press, 1971, 215-236.
9. G.Debreu, "Integration of correspondences", Proc. Fifth Berkeley Symp. on Statistics and Prob., Vol. II, Part 1, (Univ. California Press, Berkeley, 1966), 351-372.
10. R.T.Rockafellar, "Weak compactness of level sets of integral functionals", Troisième Colloque d'Analyse Fonctionelle (CBRM, Liège, 1970), H.G.Garnir (editor), 1971, 85-98.
11. M.Valadier, "Convex integrands on Suslin locally convex spaces", Travaux du Seminaire d'Analyse Convexe, Montpellier, 1973, Exposé no. 2.
12. R.T.Rockafellar, Convex Analysis, Princeton University Press, 1970.
13. I.Ekeland and R.Temam, Analyse Convexe et Problèmes Variationelles, Dunod, 1974.
14. H.Attouch, "Mesurabilité et monotonie", thèse, Paris-Orsay, 1975.
15. R.T.Rockafellar, "Existence theorems for general control problems of Bolza and Lagrange", Advances in Math. 15 (1975), 312-333.
16. R.T.Rockafellar and R.Wets, "Nonanticipativity and L^1-martingales in stochastic optimization problems", Math. Programming Studies, 5 (1975).
17. I.V.Evstigneev, "Measurable selection and dynamic programming", Math. of O.R. 1 (1976).
18. J.Lindenstrauss, "A short proof of Liapunov's convexity theorem", J.Math.Mech. 15 (1966), 971-972.
19. K.Yosida and E. Hewitt, "Finitely additive measures", Trans.Amer. Math.Soc. 72 (1952), 46-66.
20. V.L.Levin, "Lebesgue decomposition for functionals on the vector-function space L_X^∞," Funkt. Anal. + Appl. 8 (4)(1974), 48-53.
21. R.T.Rockafellar, "Level sets and continuity of conjugate convex functions", Trans. Amer. Math. Soc. 123 (1966), 46-63.
22. J.J.Moreau, "Sur la polaire d'une fonctionelle sémicontinue supérieurement", C.R.Acad.Sci. Paris 258 (1964), 1128-1130.
23. C.Castaing, "Quelques resultats de compacité liés a l'intégration", C.R.Acad.Sci. Paris 270 (1970), 1732-1735.
24. C.Castaing, "Intégrandes convexes duales", Travaux du Séminaire d'Analyse Convexe, Montpellier, 1973, Exposé no. 6.
25. M.Valadier, "Contribution a l'analyse convexe", Thèse, Paris, 1970.
26. H.Berliocchi and J.M.Lasry, "Intégrandes normales et mesures parametrées en calcul des variations", Bull.Soc.Math. France, 1974.

27. P.Clauzure, "Quelques proprietés des espaces de Köthe gén ralisés", Travaux du Séminaire d'Analyse Convexe, Montpellier, 1973, Exposé no. 4.
28. J.M.Bismut, "Intégrales convexes et probabilité", J.Math.Anal. Appl. 43 (1973), 639-673.
29. Daniel H. Wagner, "Survey of measurable selection theorems", manuscript of 80 pp. (authors address: Station Square 1, Paoli, Pennsylvania 19301, USA)
30. C. Delode, O. Arino and J.P.Penot, "Champs mesurables et multi-sections," manuscript of 37 pp. (Penot's address: Departement de Mathematiques, Boite Postale 523, 64010 Pau-Universite, France)
31. V.L.Levin, "Convex integral functionals and the theory of liftings", Uspekhi Mat. Nauk 31(2) (1975), 115-178. (For English translation, see Russian Math. Surveys (1975).)
32. R.T.Rockafellar, "Existence and duality theorems for convex problems of Bolza", Trans. A.M.S. 159 (1971), 1-40.

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